

PROJECTIVE REPRESENTATIONS OF THE INHOMOGENEOUS HAMILTON GROUP: NONINERTIAL SYMMETRY IN QUANTUM MECHANICS

S G LOW, P D JARVIS, AND R CAMPOAMOR-STURBERG

ABSTRACT. Symmetries in quantum mechanics are realized by the projective representations of the Lie group as physical states are defined only up to a phase. A cornerstone theorem shows that these representations are equivalent to the unitary representations of the central extension of the group. The formulation of the inertial states of special relativistic quantum mechanics as the projective representations of the inhomogeneous Lorentz group, and its nonrelativistic limit in terms of the Galilei group, are fundamental examples. Interestingly, neither of these symmetries includes the Weyl-Heisenberg group; the hermitian representations of its algebra are the Heisenberg commutation relations that are a foundation of quantum mechanics. The Weyl-Heisenberg group is a one dimensional central extension of the abelian group and its unitary representations are therefore a particular projective representation of the abelian group of translations on phase space. A theorem involving the automorphism group shows that the maximal symmetry that leaves invariant the Heisenberg commutation relations are essentially projective representations of the inhomogeneous symplectic group. In the nonrelativistic domain, we must also have invariance of Newtonian time. This reduces the symmetry group to the inhomogeneous Hamilton group that is a local noninertial symmetry of Hamilton's equations. The projective representations of these groups are calculated using the Mackey theorems for the general case of a nonabelian normal subgroup.

1. INTRODUCTION

Projective representations are required in quantum mechanics as the physical states in quantum mechanics are rays. A ray is an equivalence class of states in a Hilbert space that are defined up to a phase. A cornerstone theorem states that any projective representation of a Lie group is equivalent to the unitary representations of the central extension of the group. The central extension of a connected¹ group is unique and simply connected. Levi's theorem states that every simply connected Lie group is equivalent to the semidirect product of a semi-simple group and a solvable normal group [1]. Mackey's theorems provide the method of determining the unitary irreducible representations of a general class of semidirect product groups [2]. We review in Section 2 these theorems that are fundamental to the application of symmetry groups in quantum mechanics and enable us to compute the projective representations of a very general class of connected Lie groups.

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¹Connected means that all elements of the group are path connected to the identity

The inertial states of special relativistic quantum mechanics are given by the projective representations of the inhomogeneous Lorentz group [3],[4]. The inhomogeneous connected Lorentz group is the semidirect product, $\mathcal{IL}(1, n) \simeq \mathcal{L}(1, n) \otimes_s \mathcal{A}(n)$ where $\mathcal{A}(n) \simeq (\mathbb{R}^n, +)$ is the solvable abelian normal subgroup and the connected semisimple group is $\mathcal{L}(1, n)^2$. Its central extension, denoted by the inverted caret, does not have an algebraic extension and is therefore equal to its universal cover, denoted by a bar, $\widetilde{\mathcal{IL}}(1, n) \simeq \overline{\mathcal{IL}}(1, n)$. For $n = 3$, this is the Poincaré group $\overline{\mathcal{IL}}(1, 3) \simeq \mathcal{SL}(2, \mathbb{C}) \otimes_s \mathcal{A}(4)$ where $\overline{\mathcal{L}}(1, 3) \simeq \mathcal{SL}(2, \mathbb{C})^3$ [5],[6]

This is a truly remarkable and beautiful application of symmetry in physics. Starting simply with the inhomogeneous Lorentz symmetry of special relativity and the quantum mechanical condition that physical states are rays in a Hilbert space, the general group theory theorems result in the Hilbert spaces of the irreducible representations of the inertial states of special relativistic quantum mechanics labeled by the eigenvalues of the hermitian representation of the Casimir operators. For massive states, these eigenvalues are mass and spin where spin takes on integral and half integral values. The half integral values associated with fermions are a consequence of the central extension that is the Poincaré group. The central extension is a direct consequence of physical states being rays that are equivalence classes of states in a Hilbert space related by a quantum phase. Thus the quantum phase leads directly to the existence of fermion states.

However, there is no mention of the Weyl-Heisenberg group; the Heisenberg commutation relations, that are fundamental to quantum mechanics, are the hermitian representation of the algebra for the unitary representations of the Weyl-Heisenberg group.

This is not a consequence of special relativity as the Weyl-Heisenberg group does not appear in the nonrelativistic formulation either. The non-relativistic Inönü-Wigner contraction of the inhomogeneous Lorentz group is given by the inhomogeneous Euclidean group $\mathcal{IE}(n) \simeq \mathcal{E}(n) \otimes_s \mathcal{A}(n+1)$. $\mathcal{E}(n)$ is the homogeneous group, $\mathcal{E}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{A}(n)$, that is the non-relativistic limit of $\mathcal{L}(1, n)$ and is parameterized by rotations and velocity [7]. The central extension of $\mathcal{IE}(n)$ admits a one parameter algebraic extension, the generator of which is mass. This algebraic central extension defines the Galilei group $\mathcal{Ga}(n)$ whose cover is the full central extension, $\widetilde{\mathcal{IE}}(n) \simeq \overline{\mathcal{Ga}}(n) \simeq \overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+2)$ [7],[8]. In this case, the group theory results in representations that include inertial states with nonzero mass and energy with momentum diagonal that are elements of a Hilbert space of square integrable functions over \mathbb{R}^n . (For $n = 3$, this is the usual 3-momentum.) Again, there is no mention of the Weyl-Heisenberg group.

Let us put aside these relativistic considerations for a moment and instead consider a phase space \mathbb{P} that has a symplectic 2-form ω with a symmetry group $Sp(2n)$. The elements of the group may depend on the location in phase space and therefore

² $\mathcal{L}(1, n)$ is the connected component of $\mathcal{O}(1, n)$.

³We note as a case where the symmetry group is not connected that this may be extended to the groups $\mathcal{IO}(1, n)$ and $\mathcal{ISO}(1, n)$ that have the discrete symmetries parity-time-reversal, $(1, P, T, PT)$ and $(1, PT)$ respectively where P is parity and T is time-reversal. These groups are not connected and have 2 and 4 components respectively. While the central extension of a group that is not connected is not necessarily unique, it turns out that, for $\mathcal{ISO}(1, n)$ it is unique and is given in terms of the $Spin(n)$ group that is the unique cover of $\mathcal{SO}(1, n)$. On the other hand, $\mathcal{O}(1, n)$ has 8 non-isomorphic covers that include the $\mathcal{Pin}^\pm(1, n)$ group. [5], [6]

the symmetry is local. Diffeomorphisms $\phi : \mathbb{P} \rightarrow \mathbb{P}$ that leave invariant the symplectic metric, $\phi^*(\omega) = \omega$, are the usual position-momentum canonical transformations of classical mechanics. Locally, the Jacobian of the diffeomorphisms must be an element of the symplectic group. If $\mathbb{P} \simeq \mathbb{R}^{2n}$, then there is also a translational symmetry and the symmetry group is $\mathcal{ISp}(2n) \simeq \mathcal{Sp}(2n) \otimes_s \mathcal{A}(2n)$.

Consider the projective representations of $\mathcal{ISp}(2n)$ that by the fundamental theorem, are the unitary representations of its central extension, $\widetilde{\mathcal{ISp}}(2n) \simeq \overline{\mathcal{Sp}}(2n) \otimes_s \mathcal{H}(n)$. The Weyl-Heisenberg group $\mathcal{H}(n)$ is a one parameter central extension of the abelian group $\mathcal{A}(2n)$. This is the Weyl-Heisenberg group for which the commutators of the hermitian representation of the algebra are precisely the Heisenberg momentum-position commutation relations. It is a direct consequence of the physical states being rays that are equivalence classes of states in the Hilbert phase up to a phase that this noncommutative structure arises⁴[9].

If one instead were to consider the projective representation of the abelian group by itself, then the central extension $\check{\mathcal{A}}(2n)$ is required. This central extension is in general $n(2n - 1)$ dimensional. It is the presence of the symplectic homogeneous group that constrains it to precisely the Weyl-Heisenberg group. In fact, an automorphism theorem (Theorem 5) that we will review in the next section, constrains a semidirect product with $\mathcal{H}(n)$ as a normal subgroup to be a group homomorphic to a subgroup of the automorphism group of the Weyl-Heisenberg group. The connected component of the automorphism group is just the inhomogeneous symplectic group with an addition conformal multiplicative term, $\text{Aut}_{\mathcal{H}(n)} \simeq \mathcal{D}\widetilde{\mathcal{Sp}}(2n)$ where $\mathcal{D}\mathcal{Sp}(2n) \simeq \mathcal{D} \otimes_s \mathcal{ISp}(2n)$ and $\mathcal{D} \simeq (\mathbb{R}^+, \times)$. One can show that $\mathcal{D}\widetilde{\mathcal{Sp}}(2n) \simeq \mathcal{D} \otimes_s \widetilde{\mathcal{ISp}}(2n)$ and therefore it is the symplectic group that constrains the central extension to be the Weyl-Heisenberg group. Furthermore, this is the maximal group that leads to the Weyl-Heisenberg group as a result of a central extension.

The above considerations also apply to extended phase space with the addition of time and energy degrees of freedom. This results in a symplectic 2-form that may be put in the form

$$\omega = \zeta_{\alpha,\beta} dz^\alpha dz^\beta = \delta_{i,j} dp^i \wedge dq^j + dt \wedge d\varepsilon.$$

As there is no relativistic line element to distinguish time, this is just a symplectic manifold with an additional two degrees of freedom. If $\mathbb{P} \simeq \mathbb{R}^{2n+2}$, the symmetry is now $\mathcal{ISp}(2n + 2)$. The maximal symmetry for which the central extension results in the Weyl-Heisenberg group is $\mathcal{D}\mathcal{Sp}(2n + 2)$ with $\mathcal{D}\widetilde{\mathcal{Sp}}(2n + 2) \simeq \mathcal{D} \otimes_s \overline{\mathcal{Sp}}(2n + 2) \otimes_s \mathcal{H}(n + 1)$. The hermitian representations of the algebra now, at least formally, include the time, energy commutation relations.

We now have the situation where we have a symmetry group whose projective representations result in a Weyl-Heisenberg group that gives us the Heisenberg commutation relations. But as mentioned, we do not have a relativistic line element to define invariant time. For special relativity, this is just the Minkowski line element left invariant by the Lorentz group. In the nonrelativistic limit, it reduces to Newtonian time

$$d\tau^2 = dt^2 - \frac{1}{c^2} dq^2 \xrightarrow{c \rightarrow \infty} dt^2$$

⁴Note the quote from Dirac at the beginning of the Discussion section.

The invariance group for dt^2 on an m dimensional space is the affine group $\mathcal{IGL}(m-1, \mathbb{R})$

*mathrmcite*Glimore2. If one also requires length to be invariant, it reduces to the inhomogeneous Euclidean group $\mathcal{IE}(m)$ that is the nonrelativistic limit of the inhomogeneous Lorentz group discussed above.

In this paper, we are going to focus on the nonrelativistic case, for which dt^2 is invariant, along with the maximal invariance group for the Heisenberg commutation relations⁵. The intersection of the affine group and the homogeneous group $\mathcal{D} \otimes_s \mathcal{Sp}(2n+2)$ is the group $\mathcal{HSp}(2n) \simeq \mathcal{Sp}(2n) \otimes_s \mathcal{H}(n)$. This group is acting on the tangent space and this Weyl-Heisenberg group is parameterized by force, velocity and power. To understand more clearly the meaning of this symmetry, consider diffeomorphisms $\phi : \mathbb{P} \rightarrow \mathbb{P}$ that leave both ω and dt^2 invariant: $\phi^*(\omega) = \omega$ and $\phi^*(dt^2) = dt^2$. Then the Jacobian of this diffeomorphism must be an element of $\mathcal{HSp}(2n)$. The semidirect group properties imply that the diffeomorphism can be written as $\phi = \varphi \circ \tilde{\varphi}$ where both φ and $\tilde{\varphi}$ are diffeomorphisms. The Jacobian of $\tilde{\varphi}$ take their values in $\mathcal{Sp}(2n)$ and are therefore just the usual canonical transformations. The Jacobian of φ take their values in $\mathcal{H}(n)$ (that is parameterized by force, velocity and power) and this can be shown to be equivalent to Hamilton's equations [11]. In fact, $\mathcal{HSp}(2n)$ is the maximal local symmetry of Hamilton's equations. Requiring also invariance of length reduces the $\mathcal{HSp}(2n)$ to the Hamilton group that is defined to be $\mathcal{Ha}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{H}(n)$.

The projective representations of $\mathcal{IHSp}(2n) \simeq \mathcal{HSp}(2n) \otimes \mathcal{A}(2n+2)$ are the unitary representations of its central extension

$$\mathcal{IHSp}(2n) \simeq \overline{\mathcal{HSp}}(2n) \otimes_s \mathcal{H}(n+1) \simeq \overline{\mathcal{Sp}}(2n) \otimes_s \mathcal{H}(n) \otimes_s \mathcal{H}(n+1)$$

The hermitian representation of the algebra of $\mathcal{H}(n+1)$ are the Heisenberg commutation relations. The group $\mathcal{H}(n)$, on the other hand, is parameterized by force, velocity and power and acts, along with $\mathcal{Sp}(2n)$, on the tangent or cotangent space of \mathbb{P} .

The projective representations of $\mathcal{IHa}(n) \simeq \mathcal{Ha}(n) \otimes \mathcal{A}(2n+2)$ are also the unitary representations of its central extension that we will show is

$$\mathcal{IHa}(n) \simeq \overline{\mathcal{Ha}}(n) \otimes_s \mathcal{H}(n+1) \simeq \overline{\mathcal{SO}}(n) \otimes_s \mathcal{H}(n) \otimes_s (\mathcal{H}(n+1) \otimes \mathcal{A}(2))$$

The above comments on $\mathcal{H}(n)$ and $\mathcal{H}(n+1)$ apply here also. The group $\overline{\mathcal{Ga}}(n)$ is an inertial subgroup of $\mathcal{IHa}(n)$. This group has both the symmetry of Galilean relativity as its inertial special case and the Weyl-Heisenberg group that leads to the Heisenberg commutation relations.

The outline of the paper is as follows. Section two presents the mathematical framework that provides the theorems that enable the projective representations of a very general class of connected Lie groups to be determined in a fully tractable manner. These theorems are fundamental to symmetry in quantum mechanics as physical states are rays in a Hilbert space and therefore require projective representations

Section 3 studies the central extension of the inhomogeneous Hamilton group. The Hamilton group is of interest as it is a symmetry of the classical nonrelativistic Hamilton's equations from which it derives its name. This local symmetry is valid for both inertial and noninertial states. The Euclidean subgroup parameterized by

⁵Comments on the relativistic case are given in the Discussion section.

rotations and velocity is the inertial subgroup. The projective representations of the inhomogeneous Euclidean subgroup result in the inertial states of nonrelativistic quantum mechanics. This motivates the study of the projective representations of the inhomogeneous Hamilton group for the more general noninertial states.

Section 4 of this paper studies the projective representations of the inhomogeneous Hamilton group, $\mathcal{IHa}(n)$ as a global symmetry. This requires us to use the full power of the theorems of Section 2, including the nonabelian normal subgroup case of the Mackey theorems. As preliminary steps we use the theorems to compute the representations of the Weyl-Heisenberg group and the Hamilton group as these are required in the full result. As the theorems are required in their general form, these calculations are illustrative of how the projective representations of a general class of groups can be computed. The physical application of these representations are then discussed.

2. MATHEMATICAL FRAMEWORK

In this section we review a set of theorems that enable us to compute the projective representations for a general class of connected Lie groups. We refer to the cited literature for full proofs as several of the theorems are deep. Our purpose here is to assemble the theorems in a form that, when applied together, provide us with tractable tools to compute this general class of representations that includes the much studied unitary⁶ and inhomogeneous Lorentz group cases. These are fundamental theorems on the application of symmetry groups in quantum mechanics.

Our notation for a semidirect product is $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ where \mathcal{N} is the normal subgroup and \mathcal{K} is the homogeneous subgroup such that $\mathcal{K} \cap \mathcal{N} = e$ and $\mathcal{G} \simeq \mathcal{N}\mathcal{K}$ ⁷. A semidirect product is right associative in the sense that $\mathcal{D} \simeq (\mathcal{A} \otimes_s \mathcal{B}) \otimes_s \mathcal{C}$ implies that $\mathcal{D} \simeq \mathcal{A} \otimes_s (\mathcal{B} \otimes_s \mathcal{C})$ and so brackets can be removed. However $\mathcal{D} \simeq \mathcal{A} \otimes_s (\mathcal{B} \otimes_s \mathcal{C})$ does not necessarily imply $\mathcal{D} \simeq (\mathcal{A} \otimes_s \mathcal{B}) \otimes_s \mathcal{C}$ as \mathcal{B} is not necessarily a normal subgroup of \mathcal{A} .

Definition 1. An algebraic central extension of a Lie algebra g is the Lie algebra \tilde{g} that satisfies the following short exact sequence where z is the maximal abelian algebra that is central in \tilde{g} ,

$$\mathbf{0} \rightarrow z \rightarrow \tilde{g} \rightarrow g \rightarrow \mathbf{0}. \quad (1)$$

where $\mathbf{0}$ is the trivial algebra. Suppose $\{X_a\}$ is a basis of the Lie algebra g with commutation relations $[X_a, X_b] = c_{a,b}^c X_c$, $a, b = 1, \dots, r$. Then an algebraic central extension is a maximal set of central abelian generators $\{A_\alpha\}$, where $\alpha, \beta, \dots = 1, \dots, m$, such that

$$[A_\alpha, A_\beta] = 0, \quad [X_a, A_\alpha] = 0, \quad [X_a, X_b] = c_{a,b}^c X_c + c_{a,b}^\alpha A_\alpha \quad (2)$$

The basis $\{X_a, A_\alpha\}$ of the centrally extended Lie algebra must also satisfy the Jacobi identities. The Jacobi identities constrain the admissible central extensions

⁶ $SU(n) \simeq \widetilde{SU}(n)$ and is semisimple so the theorems are trivial in this case. $\mathcal{IL}(1, n)$ has an abelian normal subgroup for which the theorems simplify. As these are the most studied groups in physics, these more general theorems generally have not received the attention they deserve as the key theorems of symmetry in quantum mechanics

⁷The notation for a semidirect product varies with many authors placing the normal subgroup on the left and the reader should be sure to check conventions. Note that the semidirect product differs from a direct product in that multiplication of the normal group from the right is $\mathcal{G} \simeq (\varsigma_{\mathcal{N}} \mathcal{K}) \mathcal{N}$ where $\varsigma_h : \mathcal{G} \rightarrow \mathcal{G} : g \mapsto hgh^{-1}$, $h \in \mathcal{N}$.

of the algebra. The choice $X_a \mapsto X_a + A_a$ will always satisfy these relations and this trivial case is excluded. The algebra \check{g} constructed in this manner is equivalent to the central extension of g given in Definition 1.

Definition 2. The central extension of a connected Lie group \mathcal{G} is the Lie group $\check{\mathcal{G}}$ that satisfies the following short exact sequence where \mathcal{Z} is a maximal abelian group that is central in $\check{\mathcal{G}}$

$$e \rightarrow \mathcal{Z} \rightarrow \check{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \rightarrow e. \quad (3)$$

The abelian group \mathcal{Z} may always be written as the direct product $\mathcal{Z} \simeq \mathcal{A}(m) \otimes \mathbb{A}$ of a connected continuous abelian Lie group $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$ and a discrete abelian group \mathbb{A} that may have a finite or countable dimension [12],[13].

The exact sequence may be decomposed into an exact sequence for the *topological* central extension and the *algebraic* central extension,

$$e \rightarrow \mathbb{A} \rightarrow \bar{\mathcal{G}} \xrightarrow{\pi^\circ} \mathcal{G} \rightarrow e, \quad e \rightarrow \mathcal{A}(m) \rightarrow \check{\mathcal{G}} \xrightarrow{\tilde{\pi}} \bar{\mathcal{G}} \rightarrow e. \quad (4)$$

where $\pi = \pi^\circ \circ \tilde{\pi}$. The first exact sequence defines the universal cover where $\mathbb{A} \simeq \ker \pi^\circ$ is the fundamental homotopy group. All of the groups in the second sequence are simply connected and therefore may be defined by the exponential map of the central extension of the algebra given by Definition 1. In other words, the full central extension may be computed by determining the universal covering group of the algebraic central extension.

A ray Ψ is the equivalence class of states $|\psi\rangle$ in a Hilbert space \mathbf{H} up to a phase,

$$\Psi = \{e^{i\omega} |\psi\rangle \mid \omega \in \mathbb{R}\}, \quad |\psi\rangle \in \mathbf{H} \quad (5)$$

Note that the physical quantities that are the square of the modulus depend only on the ray

$$|(\Psi_\beta, \Psi_\alpha)|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2$$

for all $|\psi_\gamma\rangle \in \Psi$.

Definition 3. A projective representation ϱ of a symmetry group \mathcal{G} is the maximal representation such that for $|\tilde{\psi}_\gamma\rangle = \varrho(g)|\psi_\gamma\rangle$, the modulus is invariant $|\langle \tilde{\psi}_\beta | \tilde{\psi}_\alpha \rangle|^2 = |\langle \psi_\beta | \psi_\alpha \rangle|^2$ for all $|\psi_\gamma\rangle, |\tilde{\psi}_\gamma\rangle \in \Psi$.

Theorem 1. (Wigner, Weinberg) Any projective representation of a Lie symmetry group \mathcal{G} on a separable Hilbert space is equivalent to a representation that is either linear and unitary or anti-linear and anti-unitary. Furthermore, if \mathcal{G} is connected, the projective representations are equivalent to a representation that is linear and unitary [14],[4].

This is the generalization of the well known theorem that the ordinary representation of any compact group is equivalent to a representation that is unitary. For a projective representation, the phase degrees of freedom of the central extension enables the equivalent linear unitary or antilinear antiunitary representation to be constructed for this much more general class of Lie groups that admit representations on separable Hilbert spaces. (A proof of the theorem is given in Appendix A of Chapter 2 of [4]). The groups that this theorem applies to include the non-compact inhomogeneous Euclidean, Lorentz, Hamilton or unitary groups that are studied in this paper.

Theorem 2. (Bargmann, Mackey) *The projective representations of a connected Lie group \mathcal{G} are equivalent to the ordinary unitary representations of its central extension $\tilde{\mathcal{G}}$ [12],[13].*

Theorem 1 states that all projective representations are equivalent to a projective representation that is unitary. A phase is the unitary representation of a central abelian subgroup. Therefore, the maximal representation is given in terms of the central extension of the group. Appendix A shows how this definition is equivalent to the formulation of a projective representation as an ordinary unitary representation that is defined up to a phase, $\varrho(\gamma')\varrho(\gamma) = e^{i\theta}\varrho(\gamma'\gamma)$ [12],[13].

Theorem 3. *Let \mathcal{G}, \mathcal{H} be Lie groups and $\pi : \mathcal{G} \rightarrow \mathcal{H}$ be a homomorphism. Then, for every unitary representation $\tilde{\varrho}$ of \mathcal{H} there exists a degenerate unitary representation ϱ of \mathcal{G} defined by $\varrho = \tilde{\varrho} \circ \pi$. Conversely, for every degenerate unitary representation of a Lie group \mathcal{G} there exists a Lie subgroup \mathcal{H} and a homomorphism $\pi : \mathcal{G} \rightarrow \mathcal{H}$ where $\ker(\pi) \neq e$ such that $\varrho = \tilde{\varrho} \circ \pi$ where $\tilde{\varrho}$ is a unitary representation of \mathcal{H} .*

Noting that a representation is a homomorphism, This theorem follows straightforwardly from the properties of homomorphisms. As a consequence, the set of degenerate representations of a group is characterized by its set of normal subgroups. A faithful representation is the case that the representation is an isomorphism.

Theorem 4. (Levi) *Any simply connected Lie group is equivalent to the semidirect product of a semisimple group and a maximal solvable normal subgroup [1].*

As the central extension of any connected group is simply connected, the problem of computing the projective representations of a group always can be reduced to computing the unitary irreducible representations of a semidirect product group with a semisimple homogeneous group and a solvable normal subgroup. The unitary irreducible representations of the semisimple groups are known and the solvable groups that we are interested in turn out to be the semidirect product of abelian groups.

Theorem 5. *Any semidirect product group $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ is a subgroup of a group homomorphic to the group of automorphisms of the normal subgroup, $\mathcal{G} \subset \text{Aut}_{\mathcal{N}}$ [1].*

This theorem places constraints on the admissible semidirect product groups that have a given normal subgroup. For example, the automorphism group of the abelian group is

$$\text{Aut}_{\mathcal{A}(m)} \simeq \mathbb{Z}_2 \otimes_s \mathcal{D} \otimes_s \overline{\mathcal{GL}}(m, \mathbb{R}) \otimes_s \mathcal{A}(m), \mathcal{D} \simeq (\mathbb{R}^+, \times), \quad (6)$$

whereas the automorphism group of the Weyl-Heisenberg group $\mathcal{H}(m)$ is [15],[16]

$$\text{Aut}_{\mathcal{H}(m)} \simeq \mathbb{Z}_2 \otimes_s \mathcal{D} \otimes_s \overline{\mathcal{Sp}}(2m) \otimes_s \mathcal{H}(m). \quad (7)$$

This means that there does not exist a semidirect product of the form $\mathcal{SO}(2m) \otimes_s \mathcal{H}(m)$ as $\mathcal{SO}(2m)$ is not a subgroup of $\mathcal{Sp}(2m)$. On the other hand, the semidirect product $\mathcal{Ha}(n) = \mathcal{SO}(m) \otimes_s \mathcal{H}(m)$ is admissible as $\mathcal{SO}(m) \subset \mathcal{Sp}(2m)$.

2.1. Mackey theorems for the representations of semidirect product groups.

The Mackey theorems are valid for a general class of topological groups but we will only require the more restricted case $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ where the group \mathcal{G} and subgroups \mathcal{K}, \mathcal{N} are smooth Lie groups. The central extension of any connected Lie group

is simply connected and therefore generally has the form of a semidirect product due to Theorem 3 (Levi)⁸. Theorem 5 further constrains the possible homogeneous groups \mathcal{K} of the semidirect product given the normal subgroup \mathcal{N} .

The first Mackey theorem is the induced representation theorem that gives a method of constructing a unitary representation of a group (that is not necessarily a semidirect product group) from a unitary representation of a closed subgroup. The second theorem gives a construction of certain representations of a certain subgroup of a semidirect product group from which the complete set of unitary irreducible representations of the group can be induced. This theorem is valid for the general case where the normal subgroup \mathcal{N} is a nonabelian group. In the special case where the normal subgroup \mathcal{N} is abelian, the theorem may be stated in a simpler form.

Theorem 6. (Mackey) Induced representation theorem. *Suppose that \mathcal{G} is a Lie group and \mathcal{H} is a Lie subgroup, $\mathcal{H} \subset \mathcal{G}$ such that $\mathbb{K} \simeq \mathcal{G}/\mathcal{H}$ is a homogeneous space with a natural projection $\pi : \mathcal{G} \rightarrow \mathbb{K}$, an invariant measure and a canonical section $\Theta : \mathbb{K} \rightarrow \mathcal{G} : k \mapsto g$ such that $\pi \circ \Theta = \text{Id}_{\mathbb{K}}$ where $\text{Id}_{\mathbb{K}}$ is the identity map on \mathbb{K} . Let ρ be a unitary representation of \mathcal{H} on the Hilbert space \mathbf{H}^ρ :*

$$\rho(h) : \mathbf{H}^\rho \rightarrow \mathbf{H}^\rho : |\varphi\rangle \mapsto |\tilde{\varphi}\rangle = \rho(h)|\varphi\rangle, \quad h \in \mathcal{H}.$$

Then a unitary representation ϱ of a Lie group \mathcal{G} on the Hilbert space \mathbf{H}^e ,

$$\varrho(g) : \mathbf{H}^e \rightarrow \mathbf{H}^e : |\psi\rangle \mapsto |\tilde{\psi}\rangle = \varrho(g)|\psi\rangle, \quad g \in \mathcal{G},$$

may be induced from the representation ρ of \mathcal{H} by defining

$$\tilde{\psi}(k) = (\varrho(g)\psi)(k) = \rho(g^\circ)\psi(g^{-1}k), \quad g^\circ = \Theta(k)^{-1}g\Theta(g^{-1}k) \quad (8)$$

where the Hilbert space on which the induced representation ϱ acts is given by $\mathbf{H}^e \simeq \mathbf{L}^2(\mathbb{K}, \mathbf{H}^\rho)$ [2], [1].

The proof is straightforward given that the section Θ exists by showing first that $g^\circ \in \ker(\pi) \simeq \mathcal{H}$ and therefore $\rho(g^\circ)$ is well defined.

Definition 4. (Little groups) Let $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$ be a semidirect product. Let $[\xi] \in U_{\mathcal{N}}$ where $U_{\mathcal{N}}$ denotes the unitary dual whose elements are equivalence classes of unitary representations of \mathcal{N} on a Hilbert space \mathbf{H}^ξ . Let ρ be a unitary representation of a subgroup $\mathcal{G}^\circ = \mathcal{K}^\circ \otimes_s \mathcal{N}$ on the Hilbert space \mathbf{H}^ξ such that $\rho|_{\mathcal{N}} = \xi$. The *little groups* are the set of maximal subgroups \mathcal{K}° such that ρ exists on the corresponding *stabilizer* $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{N}$ and satisfies the fixed point equation

$$\hat{\varsigma}_{\rho(k)}[\xi] = [\xi], \quad k \in \mathcal{K}^\circ. \quad (9)$$

In this definition the dual automorphism is defined by

$$(\hat{\varsigma}_{\rho(g)}\xi)(h) = \rho(g)\rho(h)\rho(g)^{-1} = \rho(ghg^{-1}) = \xi(\varsigma_g h) \quad (10)$$

for all $g \in \mathcal{G}^\circ$ and $h \in \mathcal{N}$. The equivalence classes of the unitary representations of \mathcal{N} are defined by

$$[\xi] = \{\hat{\varsigma}_{\xi(h)}\xi | h \in \mathcal{N}\}. \quad (11)$$

A group \mathcal{G} may have multiple little groups \mathcal{K}°_α whose intersection is the identity element only. We will generally leave the label α implicit.

⁸The semisimple or solvable group may be trivial in which case the semidirect product is trivial

Theorem 7. (Mackey) Unitary irreducible representations of semidirect products. Suppose that we have a semidirect product Lie group $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$, where \mathcal{K}, \mathcal{N} are Lie subgroups. Let ξ be the unitary irreducible representation of \mathcal{N} on the Hilbert space \mathbf{H}^ξ . Let $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{N}$ be a maximal stabilizer on which there exists a representation ρ on \mathbf{H}^ξ such that $\rho|_{\mathcal{N}} = \xi$. Let σ be a unitary irreducible representation of \mathcal{K}° on the Hilbert space \mathbf{H}^σ . Define the representation $\varrho^\circ = \sigma \otimes \rho$ that acts on the Hilbert space $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbf{H}^\xi$. Determine the complete set of stabilizers and representations ρ and little groups that satisfy these properties, that we label by $\alpha, \{(\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})_\alpha\}$. If for some member of this set $\mathcal{G}^\circ \simeq \mathcal{G}$ then for this case the representations are $(\mathcal{G}, \varrho, \mathbf{H}^\varrho) \simeq (\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})$. For the cases where the stabilizer \mathcal{G}° is a proper subgroup of \mathcal{G} then the unitary irreducible representations $(\mathcal{G}, \varrho, \mathbf{H}^\varrho)$ are the representations induced (using Theorem 6) by the representations $(\mathcal{G}^\circ, \varrho^\circ, \mathbf{H}^{\varrho^\circ})$ of the stabilizer subgroup. The complete set of unitary irreducible representations is the union of the representations $\cup_\alpha \{(\mathcal{G}, \varrho, \mathbf{H}^\varrho)_\alpha\}$ over the set of all the stabilizers and corresponding little groups [2].

This major result and its proof are due to Mackey [2]. Our focus in this paper is on applying this theorem.

2.1.1. Abelian normal subgroup. The theorem simplifies for special cases where the normal subgroup \mathcal{N} is an abelian group, $\mathcal{N} \simeq \mathcal{A}(n)$. From Theorem 5, a semidirect product with $\mathcal{A}(n)$ as a normal subgroup is a subgroup of a homomorphism of the automorphism group (6).

An abelian group has the property that its unitary irreducible representations ξ are the characters acting on the Hilbert space $\mathbf{H}^\xi \simeq \mathbb{C}$,

$$\xi(a) |\phi\rangle = e^{ia \cdot \nu} |\phi\rangle, \quad a, \nu \in \mathbb{R}^n. \quad (12)$$

The unitary irreducible representations are labeled by the ν_i that are the eigenvalues of the hermitian representation of the basis $\{A_i\}$ of the abelian Lie algebra,

$$\hat{A}_i |\phi\rangle = \xi'(A_i) |\phi\rangle = \nu_i |\phi\rangle. \quad (13)$$

The equivalence classes $[\xi] \in \mathcal{U}_{\mathcal{A}(n)}$ each have a single element $[\xi] \simeq \xi$ as, for the abelian group, the expression (11) is trivial. The representations ρ act on $\mathbf{H}^\xi \simeq \mathbb{C}$ and are one dimensional and therefore must commute with the ξ . Therefore, in equation (10), $\rho(g)\xi(h)\rho(g)^{-1} = \xi(h)$ and (9) simplifies to

$$\xi(a) = \xi(\varsigma_k a) = \xi(kak^{-1}), \quad a \in \mathcal{A}(m), \quad k \in \mathcal{K}^\circ. \quad (14)$$

Theorem 8. (Mackey) Unitary irreducible representations of a semidirect product with an abelian normal subgroup. Suppose that we have a semidirect product group $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{A}$ where \mathcal{A} is abelian. Let ξ be the unitary irreducible representation (that are the characters) of \mathcal{A} on $\mathbf{H}^\xi \simeq \mathbb{C}$. Let $\mathcal{K}^\circ \subseteq \mathcal{K}$ be a Little group defined by (14) with the corresponding stabilizers $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{A}$. Let σ be the unitary irreducible representations of \mathcal{K}° on the Hilbert space \mathbf{H}^σ . Define the representation $\varrho^\circ = \sigma \otimes \xi$ of the stabilizer that acts on the Hilbert space $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbb{C}$. The theorem then proceeds as in the case of the general Theorem 7.

3. THE CENTRAL EXTENSION OF THE INHOMOGENEOUS HAMILTON GROUP

Consider a $2n + 2$ dimensional extended phase space manifold $\mathbb{P} \simeq \mathbb{R}^{2n+2}$ with coordinates $(q^i, p^i, t, \varepsilon)$, $i, j, \dots = 1, \dots, n$, that respectively are interpreted as position, momentum, time and energy degrees of freedom. Assume that these are global canonical coordinates in which a symplectic two form is $\omega = -d\varepsilon \wedge dt + \delta_{i,j} dp^i \wedge dq^j$ and the invariant Newtonian time element is dt^2 . $\mathcal{HSp}(2n) = \mathcal{Sp}(2n) \otimes_s \mathcal{H}(n+1)$ is the connected symmetry group that leaves both ω and dt^2 invariant. That is, a diffeomorphism $\varphi : \mathbb{P} \rightarrow \mathbb{P}$ for which the pullbacks satisfy $\varphi^*(\omega) = \omega$ and $\varphi^*(dt^2) = dt^2$ have Jacobians that are elements of the local symmetry group $\mathcal{HSp}(2n)$. This can be shown to be equivalent to the diffeomorphisms satisfying Hamilton's equations [11]. The subgroup of $\mathcal{HSp}(2n)$ that leaves invariant both dq^2 and dp^2 is the Hamilton group,

$$\mathcal{Ha}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{H}(n). \quad (15)$$

The Weyl-Heisenberg group is the semidirect product of two abelian groups,

$$\mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1), \quad (16)$$

that is a simply connected solvable group. In this case, it is parameterized by velocity v , force f and power r . It is a one parameter central extension of $\mathcal{A}(2n)^9$. As the extended phase space is $\mathbb{P} \simeq \mathbb{R}^{2n+2}$, it is also invariant under the abelian translation group $\mathcal{A}(2n+2)$. The full symmetry group is the inhomogeneous Hamilton group

$$\mathcal{IHa}(n) \simeq \mathcal{Ha}(n) \otimes_s \mathcal{A}(2n+2). \quad (17)$$

The projective representations of the inhomogeneous Hamilton group are given by the unitary representations of its central extension. The central extension has been determined in [16] to be $\mathcal{I}\widetilde{\mathcal{H}}a(n) \simeq \overline{\mathcal{QHa}}(n)$ where $\mathcal{QHa}(n)$ is the *quantum mechanical* Hamilton group that is defined to be the 3 parameter algebraic central extension of $\mathcal{IHa}(n)$,

$$\mathcal{QHa}(n) = \mathcal{Ha}(n) \otimes_s (\mathcal{H}(n+1) \otimes \mathcal{A}(2)). \quad (18)$$

In this semidirect product form, the homogeneous group \mathcal{K} is the Hamilton group and the normal subgroup is the solvable group $\mathcal{N} \simeq \mathcal{H}(n+1) \otimes \mathcal{A}(2)$. We will see shortly that the hermitian representation of the algebra corresponding to the unitary representations of the Weyl-Heisenberg subgroup $\mathcal{H}(n+1)$ are precisely the Heisenberg commutation relations for momentum and position and energy and time. Expanding the Hamilton group with (15) and using the right associativity of the semidirect product, the full central extension may be written as

$$\mathcal{I}\widetilde{\mathcal{H}}a(n) \simeq \overline{\mathcal{QHa}}(n) \simeq \overline{\mathcal{SO}}(n) \otimes_s \mathcal{H}(n) \otimes_s (\mathcal{H}(n+1) \otimes \mathcal{A}(2)). \quad (19)$$

This satisfies Levi's Theorem 4 where the simply connected semisimple group is $\mathcal{K} \simeq \overline{\mathcal{SO}}(n)$ and the simply connected maximal solvable normal subgroup is $\mathcal{N} \simeq \mathcal{H}(n) \otimes_s (\mathcal{H}(n+1) \otimes \mathcal{A}(2))$.

$\mathcal{QHa}(n)$ is a matrix group with elements realized by the matrix given in Appendix B (120). Using that parameterization, the group product for elements Γ may

⁹This is not the most general central extension of $\mathcal{A}(2n)$. The most general extension is $n(2n-1)$ dimensional.

be written as

$$\begin{aligned} \Gamma(R'', v'', f'', r'', q'', t'', p'', \varepsilon'', \iota'', s'', u'') \\ = \Gamma(R', v', f', r', q', t', p', \varepsilon', \iota', s', u') \Gamma(R, v, f, r, q, t, p, \varepsilon, \iota, s, u), \end{aligned} \quad (20)$$

where $v, f, p, q \in \mathbb{R}^n$ and $r, t, \varepsilon, \iota, s, u \in \mathbb{R}$, and $R \in \mathcal{SO}(n)$ is an $n \times n$ real matrix and¹⁰

$$\begin{aligned} R'' &= R'R, & v'' &= v' + R'v, & r'' &= r' + r + v' \cdot R'f - f' \cdot R'v, \\ t'' &= t' + t, & f'' &= f' + R'f, & \varepsilon'' &= \varepsilon' + \varepsilon + v' \cdot R'p - f' \cdot R'q + r't, \\ q'' &= q' + R'q + v't, & s'' &= s + s' + v' \cdot R'q + \frac{1}{2}t' v'^2, \\ p'' &= p' + R'p + f't, & u'' &= u + u' + f' \cdot R'p + \frac{1}{2}t' f'^2, \\ \iota'' &= \iota + \iota' + \frac{1}{2}((e' + q' \cdot f' - p' \cdot v' - r't')t - et' \\ &\quad - (p' - t'f') \cdot R'q + (q' - t'v') \cdot R'p). \end{aligned}$$

The inverse elements of the group are

$$\Gamma^{-1}(R, v, f, r, q, t, p, \varepsilon, \iota, s, u) = \Gamma(R', v', f', r', q', t', p', \varepsilon', \iota', s', u') \quad (21)$$

where

$$\begin{aligned} R' &= R^{-1}, & t' &= -t, & \varepsilon' &= -\varepsilon + v \cdot p - f \cdot q + rt, \\ v' &= -R^{-1}v, & q' &= -R^{-1}q + tR^{-1}v, & s' &= -s + v \cdot q - \frac{1}{2}tv^2, \\ f' &= -R^{-1}f, & p' &= -R^{-1}p + tR^{-1}f, & u' &= -u + f \cdot p - \frac{1}{2}tf^2, \\ \iota' &= -\iota, & r' &= -r. \end{aligned}$$

Consider the semidirect product form given in (18). The elements of the subgroups are

$$\begin{aligned} N(q, t, p, \varepsilon, \iota, s, u) &= \Gamma(1, 0, 0, 0, q, t, p, \varepsilon, \iota, s, u) \in \mathcal{N} \simeq \mathcal{H}(n+1) \otimes \mathcal{A}(2), \\ K(R, v, f, r) &= \Gamma(R, v, f, r, 0, 0, 0, 0, 0, 0, 0) \in \mathcal{K} \simeq \mathcal{Ha}(n), \end{aligned} \quad (22)$$

and

$$\begin{aligned} \Upsilon(q, t, p, \varepsilon, \iota) &\simeq N(q, t, p, \varepsilon, \iota, 0, 0) \in \mathcal{H}(n+1), \\ R &\simeq K(R, 0, 0, 0) \in \mathcal{SO}(n), \\ A(s, u) &\simeq N(0, 0, 0, 0, 0, 0, s, u) \in \mathcal{A}(2), \\ \tilde{\Upsilon}(v, f, r) &\simeq K(1, v, f, r) \in \mathcal{H}(n). \end{aligned} \quad (23)$$

The group products and inverses for these subgroups follow directly from (20) and (21). The properties of the semidirect product then implies that

$$\begin{aligned} \Gamma(R, v, q, t, p, \varepsilon, \iota, s, u) &= N(q, t, p, \varepsilon, \iota, s, u) K(R, v) \\ &= \Upsilon(q, t, p, \varepsilon, \iota) A(s, u) \tilde{\Upsilon}(v, f, r) R. \end{aligned} \quad (24)$$

This can be verified explicitly using the group product (20).

¹⁰The dot denotes the inner product $a \cdot b = a^t b = \delta_{i,j} a^i b^j$. The t denotes transpose for the matrix notation. Matrix multiplication is implicit.

3.1. Lie algebra of $\mathcal{QHa}(n)$. A general element of the algebra of $\mathcal{QHa}(n)$ may be written as

$$Z = \alpha^{i,j} J_{i,j} + v^i G_i + f^i F_i + rR + \frac{1}{\hbar} (q^i P_i + tE + p^i Q_i + \varepsilon T) + sM + uA + \iota I. \quad (25)$$

The $\alpha^{i,j}$ are the $n(n-1)/2$ dimensionless rotation angles, v^i has dimensions of velocity, f^i force, r power, q^i position, t time, p^i momentum and ε has dimensions of energy. Next, consider the parameters of the central extension. The parameter s has dimensions of mass⁻¹, u has dimensions of tension and ι is dimensionless. (\hbar is Planck's constant with the dimensions of action.) The generators have dimensions so that a general element Z is dimensionless. That is, $J_{i,j}$ is dimensionless, G_i, F_i, R have dimensions of velocity⁻¹, force⁻¹ and power⁻¹ respectively and P_i, E, Q_i, T have dimensions of momentum, energy, position and time respectively. The central generators M, A have dimensions of mass and reciprocal tension and I is dimensionless.

The algebra of the Hamilton group is [16]:

$$\begin{aligned} [J_{i,j}, J_{k,l}] &= \delta_{i,l} J_{j,k} + \delta_{j,k} J_{i,l} - \delta_{j,l} J_{i,k} - \delta_{i,k} J_{j,l}, & [G_i, F_k] &= \delta_{i,k} R, \\ [J_{i,j}, G_k] &= \delta_{j,k} G_i - \delta_{i,k} G_j, & [J_{i,j}, F_k] &= \delta_{j,k} F_i - \delta_{i,k} F_j. \end{aligned} \quad (26)$$

The inhomogeneous Hamilton group $\mathcal{IHa}(n)$ requires the additional nonzero commutation relations:

$$\begin{aligned} [J_{i,j}, Q_k] &= \delta_{j,k} Q_i - \delta_{i,k} Q_j, & [G_i, Q_k] &= \delta_{i,k} T, & [G_i, E] &= P_i, \\ [J_{i,j}, P_k] &= \delta_{j,k} P_i - \delta_{i,k} P_j, & [F_i, P_k] &= -\delta_{i,k} T, & [F_i, E] &= Q_i, \\ [R, E] &= 2T. \end{aligned} \quad (27)$$

The above relations are the algebra for $\mathcal{IHa}(n)$. T is a central generator as it commutes with all the generators. Classically, all observers related by the inhomogeneous Hamilton group have an invariant definition of time. The central extension $\mathcal{QHa}(n)$ requires the additional nonzero commutation relations

$$[P_i, Q_k] = \hbar \delta_{i,k} I, \quad [E, T] = -\hbar I, \quad [G_i, P_k] = \delta_{i,k} M, \quad [F_i, Q_k] = \delta_{i,k} A. \quad (28)$$

I, M, A are central generators as they commute with all the other generators. T is no longer central due to the nonzero $[E, T]$ commutation relation.

3.2. Subgroups of $\overline{\mathcal{QHa}}(n)$.

3.2.1. Weyl-Heisenberg. The quantum Hamilton group has several Weyl-Heisenberg subgroups. One subgroup is $\mathcal{H}(n+1)$ that has an algebra spanned by $\{P_i, Q_i, E, T, I\}$. The hermitian representation of the algebra corresponding to the unitary representations of this group, as we will see in Section 4, are the usual Heisenberg commutation relations for position-time and energy-momentum.

Another Weyl-Heisenberg $\mathcal{H}(n)$ subgroup is the subgroup that is generated by $\{F_i, G_i, R\}$. Furthermore, there are two Weyl-Heisenberg subgroups with an algebra generated by $\{G_i, P_i, M\}$ and $\{F_i, Q_i, A\}$ respectively. Finally, there are the two additional Weyl-Heisenberg subalgebras generated by $\{G_i, Q_k, T\}$ and $\{F_i, P_i, T\}$. (Note that $\{I, M, A\}$ are central generators of the algebra of the full $\mathcal{QHa}(n)$ group whereas T and R are not.)

3.2.2. *Hamilton.* The cover of the Hamilton group $\overline{\mathcal{H}a}(n)$ is a subgroup of $\overline{\mathcal{Q}Ha}(n)$ with an algebra with the general element

$$Z = \alpha^{i,j} J_{i,j} + v^i G_i + f^i F_i + rR. \quad (29)$$

It may be written as a semidirect product in either of the forms

$$\overline{\mathcal{H}a}(n) \simeq \overline{\mathcal{SO}}(n) \otimes_s \mathcal{H}(n) \simeq \overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+1), \quad \overline{\mathcal{E}}(n) \simeq \overline{\mathcal{SO}}(n) \otimes_s \mathcal{A}(n). \quad (30)$$

The cover of the special orthogonal group $\overline{\mathcal{SO}}(n)$ has generators $\{J_{i,j}\}$. In the first semidirect product form, the Weyl-Heisenberg subgroup has generators $\{G_i, F_i, R\}$ noted in the previous section. Alternatively, the Weyl-Heisenberg group may be expanded as the semidirect product of abelian groups given in (16). There are two choices for its normal subgroup $\mathcal{A}(n+1)$. The first case, the $\mathcal{A}(n+1)$ subgroup has an algebra spanned by $\{F_i, R\}$ and the second case it is spanned by $\{G_i, R\}$. The corresponding generators for the homogenous group $\overline{\mathcal{E}}(n)$ are $\{J_{i,j}, G_i\}$ for the first case and $\{J_{i,j}, F_i\}$ for the second.

3.2.3. *Galilei.* The cover of the Galilei group may be written in several different semidirect product forms

$$\begin{aligned} \widetilde{\mathcal{IE}}(n) \simeq \overline{\mathcal{Ga}}(n) &\simeq \overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+2) \\ &\simeq (\mathcal{A}(1) \otimes \overline{\mathcal{SO}}(n)) \otimes_s \mathcal{H}(n) \simeq \mathcal{A}(1) \otimes_s \overline{\mathcal{H}a}(n) \\ &\simeq \overline{\mathcal{SO}}(n) \otimes_s \mathcal{A}(1) \otimes_s \mathcal{H}(n). \end{aligned} \quad (31)$$

The last form is the Levi decomposition where the simply connected semisimple homogeneous group is $\mathcal{K} \simeq \overline{\mathcal{SO}}(n)$ and the simply connected solvable normal subgroup is $\mathcal{N} \simeq \mathcal{A}(1) \otimes_s \mathcal{H}(n)$.

There are two distinct $\overline{\mathcal{Ga}}(n)$ groups that are subgroups of $\overline{\mathcal{Q}Ha}(n)$ with general elements given by

$\overline{\mathcal{Q}Ha}(n)$ has two distinct Galilei group subgroups $\overline{\mathcal{Ga}}(n)$. General elements of their respective Lie algebras take the form

$$Z_1 = \alpha^{i,j} J_{i,j} + v^i G_i + \frac{1}{\hbar} (q^i P_i + tE) + sM. \quad (32)$$

$$Z_2 = \alpha^{i,j} J_{i,j} + f^i F_i + \frac{1}{\hbar} (p^i Q_i + tE) + uA. \quad (33)$$

Elements of the form Z_1 generate the the usual *physical* Galilei group. For this subgroup, written as the semidirect product $\overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+2)$, the subgroup $\overline{\mathcal{E}}(n)$ has generators $\{J_{i,j}, G_i\}$ and the $\mathcal{A}(n+2)$ has generators $\{P_i, E, M\}$. Written as the semidirect product $(\mathcal{A}(1) \otimes \overline{\mathcal{SO}}(n)) \otimes_s \mathcal{H}(n)$, the subgroup $\mathcal{A}(1) \otimes \overline{\mathcal{SO}}(n)$ has generators $\{E, J_{i,j}\}$ and the $\mathcal{H}(n)$ has generators $\{G_i, P_i, M\}$.

Alternatively, elements of the form Z_2 general a different Galilei subgroup $\overline{\mathcal{Ga}}(n)$. Again, this group may be written as the semidirect product $\overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+2)$, where the $\overline{\mathcal{E}}(n)$ has generators $\{J_{i,j}, F_i\}$ and the $\mathcal{A}(n+2)$ has generators $\{Q_i, T, A\}$. Written as the semidirect product $(\mathcal{A}(1) \otimes \overline{\mathcal{SO}}(n)) \otimes_s \mathcal{H}(n)$, the $\mathcal{A}(1) \otimes \overline{\mathcal{SO}}(n)$ has generators $\{E, J_{i,j}\}$ and the $\mathcal{H}(n)$ has generators $\{F_i, Q_i, A\}$.

3.3. **Casimir invariants.** Any element in the center $\mathbf{z}(g)$ of the enveloping algebra $\mathbf{e}(g)$ is invariant of the algebra g . The Casimir invariants form a basis in the sense that any element of the center $\mathbf{z}(g)$ may be written as a polynomial of the basis

elements. The number of N_c Casimir invariants of $\mathcal{QHa}(n)$ and of its Galilei and Hamilton subgroups is given in the following table:¹¹

	1	2	3	4	n
$\mathcal{H}(n)$	1	1	1	1	1
$\mathcal{Ha}(n)$	1	2	2	3	$\left\lfloor \frac{n}{2} \right\rfloor + 1$
$\mathcal{Ga}(n)$	2	2	3	3	$\left\lfloor \frac{n}{2} \right\rfloor + 2$
$\mathcal{QHa}(n)$	4	4	5	5	$\left\lfloor \frac{n}{2} \right\rfloor + 4$

The Casimir invariants of $\mathcal{QHa}(3)$ have been determined to be [16],[17]

$$C_1 = I, C_2 = M, C_3 = A, C_4 = T^2 - IR, C_5 = B_{i,j}B_{i,j}, \quad (34)$$

The $B_{i,j}$ are defined, using the auxiliary invariant C as

$$B_{i,j} = CJ_{i,j} + D_{i,j}, \quad C = -AM + T^2 - IR. \quad (35)$$

The $D_{i,j}$ are given by

$$D_{i,j} = AD_{i,j}^1 + MD_{i,j}^2 + RD_{i,j}^3 + ID_{i,j}^4 + T(D_{i,j}^5 + D_{i,j}^6), \quad (36)$$

where

$$\begin{aligned} D_{i,j}^1 &= G_j P_i - G_i P_j, & D_{i,j}^3 &= P_i Q_j - P_j Q_i, & D_{i,j}^5 &= F_i P_j - F_j P_i, \\ D_{i,j}^2 &= F_j Q_i - F_i Q_j, & D_{i,j}^4 &= F_i G_j - F_j G_i, & D_{i,j}^6 &= G_i Q_j - G_j Q_i. \end{aligned} \quad (37)$$

The Casimir invariant of $\mathcal{H}(n)$ is the central element I . The Casimir invariants of $\mathcal{Ha}(3)$ subgroup are

$$C_1 = R, \quad C_2 = B_{i,j}B_{i,j}, \quad B_{i,j} = RJ_{i,j} + F_j G_i - F_i G_j \quad (38)$$

The Casimir invariants of the two $\mathcal{Ga}(3)$ subgroups are

$$\begin{aligned} C_1 &= M, & C_2 &= 2ME - P^2, \\ C_3 &= B_{i,j}B_{i,j}, & B_{i,j} &= MJ_{i,j} - G_j P_i + G_i P_j, \end{aligned} \quad (39)$$

and

$$\begin{aligned} C_1 &= A, & C_2 &= 2AE - Q^2, \\ C_3 &= B_{i,j}B_{i,j}, & B_{i,j} &= AJ_{i,j} - F_j Q_i + F_i Q_j. \end{aligned} \quad (40)$$

3.4. Homomorphisms. The groups homomorphic to $\overline{\mathcal{QHa}}(n)$ also appear in the representation theory as degenerate representations due to Theorem 3. The homomorphisms are characterized by the normal subgroups of the group that are the kernels of the homomorphisms. Appendix C summarizes the homomorphic groups for the Weyl-Heisenberg, Hamilton, Galilei and the quantum mechanical Hamilton groups.

¹¹The cover of these groups, $\overline{\mathcal{Ha}}(n)$, $\overline{\mathcal{Ga}}(n)$ and $\overline{\mathcal{QHa}}(n)$ have the same algebra and therefore the same Casimir invariants.

4. PROJECTIVE REPRESENTATIONS OF THE INHOMOGENEOUS HAMILTON GROUP

The Mackey theorems may be used to compute the unitary irreducible representations of the central extension of the inhomogeneous Hamilton group that are required by quantum mechanics. We will need the unitary irreducible representations of the Weyl-Heisenberg and Hamilton group in this calculation and we therefore review these first.

A unitary representation $\varrho : \mathcal{G} \rightarrow U(\mathbf{H}) : g \mapsto \varrho(g)$ satisfies the unitary condition $\varrho(g)^{-1} = \varrho(g)^\dagger$. For a simply connected group, the group elements are $g = e^X$. The representation is $\varrho(g) = e^{\varrho'(X)}$ and the unitary condition requires $\varrho'(X) = -\varrho'(X)^\dagger$ and therefore the representation of the algebra is anti-hermitian. The standard physics convention is to use hermitian operators by defining $X \mapsto -iX$ so that $\varrho'(X) = \varrho'(X)^\dagger$. This requires an i to also be inserted in the algebra commutation relations. That is, if $[X_i, X_j] = c_{i,j}^k X_k$ then the hermitian representation $\hat{X}_i = \varrho'(X_i)$ satisfies the commutation relations

$$[\hat{X}_i, \hat{X}_j] = i c_{i,j}^k \hat{X}_k. \quad (41)$$

4.1. Unitary irreducible representations of the Weyl-Heisenberg group.

The unitary representations of the Weyl-Heisenberg group

$$\mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1), \quad (42)$$

may be determined using the abelian Mackey Theorem 8 [18],[19]. The Weyl-Heisenberg group product and inverse for elements $\Upsilon(a, b, \iota)$ is the special case (23) of (20),

$$\begin{aligned} \Upsilon(a', b', \iota') \Upsilon(a, b, \iota) &= \Upsilon(a' + a, b' + b, \iota' + \iota + \frac{1}{2}(a' \cdot b - b' \cdot a)), \\ \Upsilon^{-1}(a, b, \iota) &= \Upsilon(-a, -b, -\iota). \end{aligned} \quad (43)$$

The inner automorphisms are

$$\varsigma_{\Upsilon(a', b', \iota')} \Upsilon(a, b, \iota) = \Upsilon(a, b, \iota + a' \cdot b - b' \cdot a). \quad (44)$$

Elements of the normal subgroup $\mathcal{A}(n+1)$ may be taken to be either $\Upsilon(a, 0, \iota)$ or $\Upsilon(0, b, \iota)$. The corresponding elements of the homogeneous group $\mathcal{A}(n)$ are $\Upsilon(0, b, 0)$ or $\Upsilon(a, 0, 0)$. A general element of the Weyl-Heisenberg group may be written as

$$\Upsilon(a, b, \iota) = \Upsilon(a, 0, \iota - \frac{1}{2}a \cdot b) \Upsilon(0, b, 0) = \Upsilon(0, b, \iota + \frac{1}{2}a \cdot b) \Upsilon(a, 0, 0). \quad (45)$$

Noting that

$$\zeta_\pm : \mathcal{H}(n) \rightarrow \mathcal{H}(n) : \Upsilon(a, b, \iota) \mapsto \Upsilon(a, b, \iota \pm \frac{1}{2}a \cdot b), \quad (46)$$

is a group isomorphism, it may be straightforwardly shown that either choice of the normal subgroup elements in the expressions in (45) satisfy the properties to be the semidirect product (42).

4.1.1. Mackey abelian semidirect product theorem. The Mackey Theorem 8 may now be applied. We choose the normal subgroup with elements $\Upsilon(a, 0, \iota) \in \mathcal{A}(n+1)$. The unitary irreducible representations ξ of the abelian normal subgroup are the phases acting on the Hilbert space $\mathbf{H}^\xi = \mathbb{C}$

$$\xi(\Upsilon(a, 0, \iota))|\phi\rangle = e^{i(a^i \hat{A}_i + \iota \hat{I})} |\phi\rangle = e^{i(a \cdot \alpha + \iota \lambda)} |\phi\rangle, \quad |\phi\rangle \in \mathbb{C}. \quad (47)$$

The hermitian representation of the algebra has the eigenvalues that are given by

$$\hat{A}_i |\phi\rangle = \xi'(A_i) |\phi\rangle = \alpha_i |\phi\rangle, \quad \hat{I} |\phi\rangle = \xi'(I) |\phi\rangle = \lambda |\phi\rangle, \quad (48)$$

where $\alpha \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The characters $\xi_{\alpha,\lambda}$ are parameterized by the eigenvalues α, λ and the equivalence classes that are elements of the unitary dual, $[\xi_{\alpha,\lambda}] \in U_{\mathcal{A}(n+1)} \simeq \mathbb{R}^{n+1}$. Each equivalence class has the single element $[\xi_{\alpha,\lambda}] = \xi_{\alpha,\lambda}$.

The action of the elements $\Upsilon(0, b, 0) \in \mathcal{A}(n)$ of the homogeneous group on these representations is given by the dual automorphisms

$$\begin{aligned} (\hat{\Upsilon}_{(0,b,0)} \xi_{\alpha,\lambda}) (\Upsilon(a, 0, \iota)) |\phi\rangle &= \xi_{\alpha,\lambda} (\varsigma_{\Upsilon(0,b,0)} \Upsilon(a, 0, \iota)) |\phi\rangle \\ &= \xi_{\alpha,\lambda} (\Upsilon(a, 0, \iota - a \cdot b)) |\phi\rangle \\ &= e^{i(a \cdot (\alpha - \lambda b) + \iota \lambda)} |\phi\rangle \\ &= \xi_{\alpha - \lambda b, \lambda} (\Upsilon(a, 0, \iota)) |\phi\rangle. \end{aligned} \quad (49)$$

Therefore, the little group is the set of $\Upsilon(0, b, 0) \in \mathcal{K}^\circ$ that satisfy the fixed point equation (14),

$$\hat{\Upsilon}_{(0,b,0)} \xi_{\alpha,\lambda} = \xi_{\alpha - \lambda b, \lambda} = \xi_{\alpha,\lambda}. \quad (50)$$

The solution of the fixed point condition requires that $\alpha - \lambda b \equiv \alpha$. The $\lambda = 0$ solution for which the little group is $\mathcal{A}(n)$ is the degenerate case corresponding to the homomorphism $\mathcal{H}(n) \rightarrow \mathcal{A}(2n)$ with kernel $\mathcal{A}(1)$ (See Appendix C, (123)). This is just the abelian group that is not considered further here. The faithful representation with $\lambda \neq 0$ requires $b = 0$, and therefore has the trivial little group $\mathcal{K}^\circ \simeq e \simeq \{\Upsilon(0, 0, 0)\}$. The stabilizer is $\mathcal{G}^\circ \simeq \mathcal{A}(n+1)$. The orbits are

$$\mathcal{O}_\lambda = \{\hat{\Upsilon}_{(0,b,0)} [\xi_{\alpha,\lambda}] | b \in \mathbb{R}^n\} = \{\xi_{b,\lambda} | b \in \mathbb{R}^n\}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (51)$$

All representations in the orbit are equivalent for the determination of the semidirect product unitary irreducible representations. A convenient representative of the equivalence class is $\xi_{0,\lambda}$. The unitary representations σ of the trivial little group are trivial and therefore the representations of the stabilizer are just $\varrho^\circ = \xi_{0,\lambda}$. The Hilbert space \mathbf{H}^σ is also trivial and therefore the Hilbert space of the stabilizer is $\mathbf{H}^{\varrho^\circ} = \mathbf{H}^\sigma \otimes \mathbf{H}^\xi \simeq \mathbb{C}$.

4.1.2. Mackey induction. The final step is to apply the Mackey induction theorem to determine the faithful unitary irreducible representations of the full $\mathcal{H}(n)$ group. The induction requires the definition of the symmetric space

$$\mathbb{K} = \mathcal{G}/\mathcal{G}^\circ = \mathcal{H}(n)/\mathcal{A}(n+1) \simeq \mathcal{A}(n) \simeq \mathbb{R}^n, \quad (52)$$

with the natural projection π and a section Θ

$$\begin{aligned} \pi : \mathcal{H}(n) &\rightarrow \mathbb{K} : \Upsilon(a, b, \iota) \mapsto k_b, \\ \Theta : \mathbb{K} &\rightarrow \mathcal{H}(n) : k_b \mapsto \Theta(k_b) = \Upsilon(0, b, 0). \end{aligned} \quad (53)$$

These satisfy $\pi(\Theta(k_b)) = k_b$ and so $\pi \circ \Theta = \text{Id}_{\mathbb{K}}$ as required. Using (2), an element of the Weyl-Heisenberg group $\mathcal{H}(n)$ can be written as,

$$\Upsilon(a, b, \iota) = \Upsilon(0, b, 0) \Upsilon(a, 0, \iota + \frac{1}{2} a \cdot b). \quad (54)$$

The cosets are therefore defined by

$$\begin{aligned} k_b &= \{\Upsilon(0, b, 0) \Upsilon(a, 0, \iota + \frac{1}{2} a \cdot b) | a \in \mathbb{R}^n, \iota \in \mathbb{R}\} \\ &= \{\Upsilon(0, b, 0) \mathcal{A}(n+1)\} \end{aligned} \quad (55)$$

Note that

$$\Upsilon(a, b, \iota)k_x = k_{x+b}, \quad x \in \mathbb{R}^n. \quad (56)$$

The Mackey induced representation theorem can now be applied straightforwardly. First, the Hilbert space is

$$\mathbf{H}^\varrho = \mathbf{L}^2(\mathbb{K}, H^{\varrho^\circ}) \simeq \mathbf{L}^2(\mathbb{R}^n, \mathbb{C}). \quad (57)$$

Next the Mackey induction Theorem 7 yields

$$\psi'(k_x) = (\varrho(\Upsilon(a, b, \iota))\psi) \left(\Upsilon(a, b, \iota)^{-1}k_x \right) = \varrho^\circ(\Upsilon(a^\circ, 0, \iota^\circ))\psi(k_{x-b}) \quad (58)$$

Using the Weyl-Heisenberg group product (2),

$$\begin{aligned} \Upsilon(a^\circ, b^\circ, \iota^\circ) &= \Theta(k_x)^{-1}\Upsilon(a, b, \iota)\Theta(\Upsilon(a, b, \iota)^{-1}k_x) \\ &= \Upsilon(0, -x, 0)\Upsilon(a, b, \iota)\Upsilon(0, x-b, 0) \\ &= \Upsilon(a, 0, \iota + a \cdot (x - \tfrac{1}{2}b)). \end{aligned} \quad (59)$$

We lighten notation using the isomorphism $k_x \mapsto x$. The induced representation theorem then yields

$$\begin{aligned} \psi'(x) &= \xi_{0,\lambda}(\Upsilon(a, 0, \iota + x \cdot a - \tfrac{1}{2}a \cdot b)\psi(x-b)) \\ &= e^{i\lambda(\iota + x \cdot a - \frac{1}{2}a \cdot b)}\psi(x-b). \end{aligned} \quad (60)$$

Using Taylor expansion, we can write

$$\psi(x-b) = e^{-b^i \frac{\partial}{\partial x^i}} \psi(x). \quad (61)$$

The Baker Campbell-Hausdorff formula [20] enables us to combine the exponentials

$$\psi'(x) = e^{i(\lambda\iota + \lambda a^i x_i + b^i i \frac{\partial}{\partial x^i})}\psi(x) = e^{i(a^i \hat{A}_i + b^i \hat{B}_i + \iota \hat{I})}\psi(x). \quad (62)$$

The representation of the algebra is therefore

$$\hat{I}\psi(x) = \lambda\psi(x), \quad \hat{A}_i\psi(x) = \lambda x_i\psi(x), \quad \hat{B}_i\psi(x) = i \frac{\partial}{\partial x^i}\psi(x), \quad (63)$$

that satisfies the commutation relations, $[\hat{B}_i, \hat{A}_j] = i\delta_{i,j}\hat{I}$. This analysis can also be carried out choosing $\Upsilon(0, b, \iota) \in \mathcal{A}(n+1)$ to be the elements of the normal subgroup and this yields the representation with \hat{B}_i diagonal.

4.2. Unitary irreducible representations of the Hamilton group. Again, from Theorem 2, the projective representations of the Hamilton group are equivalent to unitary representations of its cover. The Hamilton group does not admit an algebraic extension and therefore the central extension of the Hamilton group is its cover, $\widetilde{\mathcal{H}a}(n) \simeq \overline{\mathcal{H}a}(n)$. The cover of the Hamilton group may be written as a semidirect product group in the different forms given in (12).

The Mackey theorems may be used to compute the unitary irreducible representations using any of these semi-direct product forms. We use here the form with the nonabelian Weyl-Heisenberg group $\mathcal{H}(n)$ as the normal subgroup. The reader can verify that the Mackey theorem applied to the form with the abelian normal subgroup $\mathcal{A}(n+1)$ gives the same result.

For simplicity of exposition, we give the computation for the representations of $\mathcal{H}a(n)$. Appendix D shows how the representation of its cover $\overline{\mathcal{H}a}(n)$ is computed from these results.

4.2.1. *Unitary representations of the Weyl-Heisenberg normal subgroup.* We can apply the results of the previous section where we determined the unitary irreducible representations of the Weyl-Heisenberg group. In this section, the elements are $\Upsilon(v, f, r) \in \mathcal{H}(n)$ and the general element of the algebra is given in (29). Choosing $\Upsilon(v, 0, r)$ to be the elements of the normal subgroup $\mathcal{A}(n+1)$ of $\mathcal{H}(n)$, the representations from (60) are

$$\varphi'(x) = (\xi(\Upsilon(v, f, r))\varphi)(x) = e^{i\kappa(r - \frac{1}{2}f \cdot v) + i\kappa v \cdot x} \varphi(x - f), \quad (64)$$

where $f, v, x \in \mathbb{R}^n, r, \kappa \in \mathbb{R}, \kappa \neq 0$ with $\varphi \in \mathbf{H}^\xi \simeq \mathbf{L}^2(\mathbb{R}^n, \mathbb{C})$. The hermitian representation of the basis $\{F_i, G_i, R\}$ of the Weyl-Heisenberg algebra are

$$\hat{R} = \xi'(R) = \kappa, \quad \hat{G}_i = \xi'(G_i) = \kappa x_i, \quad \hat{F}_i = \xi'(F_i) = -i \frac{\partial}{\partial x^i}. \quad (65)$$

Similar expressions result with the choice of elements $\Upsilon(0, f, r)$ of the normal subgroup for which \hat{F}_i is diagonal.

4.2.2. *The ρ representation.* The next step is to determine the stabilizer \mathcal{G}° and the representation ρ . It acts on the Hilbert space \mathbf{H}^ξ and therefore the hermitian representations ρ' of the algebra of the little group must be realized in the enveloping algebra of the Weyl-Heisenberg group. The hermitian ρ' representation of the basis of the algebra of the Weyl-Heisenberg group are given by (65) as $\rho|_{\mathcal{H}(n)} = \xi$. Define the hermitian ρ' representation generators as

$$\hat{J}_{i,j} = \rho'(J_{i,j}) = \frac{1}{\kappa} (\hat{F}_i \hat{G}_j - \hat{F}_j \hat{G}_i). \quad (66)$$

The commutation relations for the generators in the ρ' representation are directly computed using (65) to be the commutation relations for the Hamilton algebra given by (26) with the i inserted for the hermitian representation as noted in (41).

This is a ρ' representation of the entire algebra of the Hamilton group and therefore the stabilizer is the Hamilton group itself. Using (22), the group action is given by

$$\rho(\mathbf{R}(\theta))\varphi(x) = e^{-\theta^{k,j}(x_k \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^k})} \psi(x) = \varphi(\mathbf{R}^{-1}(\theta)x). \quad (67)$$

Using the semidirect group property (24), $\mathbf{K}(\mathbf{R}, v, f, r) = \Upsilon(v, f, r)\mathbf{R}$, where $\Upsilon(v, f, r) \simeq \mathbf{K}(1, v, f, r)$ and $\mathbf{R} \simeq \mathbf{K}(\mathbf{R}, 0, 0, 0)$ and putting it together with (20), the full ρ representation is

$$\varphi'(x) = (\xi(\Upsilon(v, f, r))\rho(\mathbf{R})\varphi)(x) = e^{i\kappa(r - \frac{1}{2}v \cdot f + v \cdot x)} \varphi(\mathbf{R}^{-1}x - f). \quad (68)$$

As the stabilizer is the full Hamilton group (and the little group is $\mathcal{K} = \overline{\mathcal{SO}}(n)$), the Mackey theorem (Theorem 7) applies without requiring use of the induced representation Theorem 6.

4.2.3. *Nondegenerate unitary irreducible representations.* The faithful unitary irreducible representations of the Hamilton group from the application of the Mackey theorem (Theorem 7) is

$$\sigma(\mathbf{K}(\mathbf{R}, v, f, r)) = \sigma(\mathbf{R}) \otimes \rho(\mathbf{K}(\mathbf{R}, v, f, r)) = \sigma(\mathbf{R}) \otimes \xi(\Upsilon(v, f, r))\rho(\mathbf{R}). \quad (69)$$

As the stabilizer is the group itself, induction is not required. The $\sigma(\mathbf{R})$ are the ordinary unitary irreducible representations of $\overline{\mathcal{SO}}(n)$ that act on a finite dimensional

Hilbert space $\mathbf{H}^\sigma \simeq \mathbb{V}^N$. Therefore, the full representation acts on the Hilbert space

$$\mathbf{H}^\varrho = \mathbf{H}^\sigma \otimes \mathbf{H}^\xi = \mathbb{V}^N \otimes \mathbf{L}^2(\mathbb{R}^n, \mathbb{C}). \quad (70)$$

The full nondegenerate Hamilton representations are the direct product given in (27).

$$\begin{aligned} \varphi'(x) &= (\varrho(K(R, v, f, r))\varphi)(x) \\ &= (\sigma(R) \otimes \xi(\Upsilon(v, f, r))\rho(R)\varphi)(x). \end{aligned} \quad (71)$$

In particular, for $n = 3$, $\overline{\mathcal{SO}}(3) = \mathcal{SU}(2)$ with $N = 2j + 1$ and the σ representation is given in terms of the well known $D^j(R(\theta))$ representation matrices. For notation reasons, we set $x = \tilde{f}$ as it is clear that it has the meaning of force with κ having the dimensions of the reciprocal of power,

$$\varphi'_m(\tilde{f}) = D^j(R)_m^m e^{i\kappa(r - \frac{1}{2}v \cdot f + v \cdot \tilde{f})} \varphi_m(R^{-1}\tilde{f} - f). \quad (72)$$

The above representations use the choice of the normal subgroup of $\mathcal{A}(n + 1)$ of the Weyl-Heisenberg subgroup $\mathcal{H}(n)$ that is generated by $\{G_i, R\}$. These generators are diagonal in the resulting representations (65) and (72). We could equally well have chosen the normal subgroup to be generated by $\{F_i, R\}$ resulting in similar representations with these generators diagonal

$$\tilde{\varphi}'_{m'}(\tilde{v}) = D^j(R)_{m'}^m e^{i\kappa(r + \frac{1}{2}v \cdot f + f \cdot \tilde{v})} \tilde{\varphi}_m(R^{-1}\tilde{v} - v). \quad (73)$$

The degenerate cases corresponding to the homomorphisms in Appendix C (123) may be similarly computed.

4.2.4. *Casimir invariants.* The Casimir invariants for the case $n = 3$ are given in (20). A direct computation shows that $\rho'(C_2) = 0$. Therefore,

$$\varrho'(C_1) = \rho'(C_1) = \rho'(R) = \kappa, \quad \varrho'(C_2) = \rho'(R)^2 \sigma'(J^2) = \kappa^2 j(j + 1), \quad (74)$$

where $\kappa \in \mathbb{R} \setminus \{0\}$, and j is half integral. The (κ, j) label the faithful irreducible representations. Similar considerations apply for the degenerate cases.

4.3. **Unitary irreducible representations of the Galilei group.** The projective representations of the inhomogeneous Euclidean group are equivalent to the unitary representations of the cover of the Galilei group. The Galilei group may be written as a semidirect product in several different forms (31). Any of these forms may be used in the Mackey theorems to determine the unitary irreducible representations. The Mackey theorems for the form with the abelian normal subgroup $\mathcal{A}(n + 1)$ has been studied in reference [8]. We choose here to use the form where $\mathcal{N} \simeq \mathcal{H}(n)$ is the normal subgroup and the homogeneous group is $\mathcal{K} \simeq \mathcal{A}(1) \otimes \overline{\mathcal{SO}}(n)$. A general element of the Galilei group is

$$\Gamma(R, t, v, q, s) \simeq \Upsilon(v, q, s)A(t)R \quad (75)$$

where $\Upsilon(v, q, s) \in \mathcal{H}(n)$, $A(t) \in \mathcal{A}(1)$ and $R \in \overline{\mathcal{SO}}(n)$. The general element of the algebra is given in (32). Note that while the Υ are again elements of the Weyl-Heisenberg group, in this case they are parameterized by position, velocity and the parameter of the central generator that is mass.

Again, the faithful unitary irreducible representation of elements $\Upsilon(v, q, s) \in \mathcal{H}(n)$ are

$$\varphi'(\tilde{p}) = (\xi(\Upsilon(v, q, s))\varphi)(u) = e^{\frac{i}{\hbar}(\mu(s - \frac{1}{2}q \cdot v) + q \cdot \tilde{p})} \varphi(\tilde{p} - \mu v), \quad (76)$$

where $q, \tilde{p}, v \in \mathbb{R}^n$, $s, \mu \in \mathbb{R}$, $\mu \neq 0$ and $\varphi \in \mathbf{H}^\xi \simeq \mathbf{L}^2(\mathbb{R}^n, \mathbb{C})$. The isomorphism (46) enables us to re-parameterize $s - \frac{1}{2}q \cdot v \mapsto s$. The μ are the eigenvalues of the hermitian representation of the central element M of the basis $\{G_i, P_i, M\}$ of the Weyl-Heisenberg algebra. The ξ' representation of these generators, with momentum diagonal, are

$$\widehat{M} = \xi'(M) = \mu, \quad \widehat{P}_i = \xi'(P_i) = \tilde{p}_i, \quad \widehat{G}_i = \xi'(G_i) = i\mu \frac{\partial}{\partial \tilde{p}_i}. \quad (77)$$

4.3.1. *The ρ representation.* The next step is to determine the stabilizer \mathcal{G}° and the representation ρ . It acts on the Hilbert space \mathbf{H}^ξ and therefore the hermitian representations ρ' of the little group must be realized in the enveloping algebra of the Weyl-Heisenberg group [19]. The ρ' representation restricted to the normal subgroup are the ξ' representation given in (77), $\rho|_{\mathcal{H}(n)} = \xi$. The generators of the homogeneous group in the ρ' representation with momentum diagonal are

$$\widehat{J}_{i,j} = \rho'(J_{i,j}) = \frac{1}{\mu} \left(\widehat{P}_i \widehat{G}_j - \widehat{P}_j \widehat{G}_i \right), \quad \widehat{E} = \rho'(E) = \frac{1}{2\mu} \widehat{P}^2. \quad (78)$$

These satisfy the commutation relations

$$\begin{aligned} \left[\widehat{J}_{i,j}, \widehat{J}_{k,l} \right] &= i(\widehat{J}_{j,k} \delta_{i,l} + \widehat{J}_{i,l} \delta_{j,k} - \widehat{J}_{i,k} \delta_{j,l} - \widehat{J}_{j,l} \delta_{i,k}), \\ \left[\widehat{J}_{i,j}, \widehat{G}_k \right] &= i(\widehat{G}_j \delta_{i,k} - \widehat{G}_i \delta_{j,k}), \quad \left[\widehat{J}_{i,j}, \widehat{P}_k \right] = i(\widehat{P}_i \delta_{j,k} - \widehat{P}_j \delta_{i,k}), \\ \left[\widehat{G}_i, \widehat{E} \right] &= i\widehat{P}_i, \quad \left[\widehat{G}_i, \widehat{P}_k \right] = i\widehat{M} \delta_{i,k}. \end{aligned} \quad (79)$$

These are the commutation relations for the Galilei subgroup given by (26-28) for the generators $\{J_{i,j}, G_i, P_i, E, M\}$ with the i inserted for the hermitian representation as noted in (41).

4.3.2. *Nondegenerate unitary irreducible representations.* The properties of the semidirect product enable us to write the ρ representation as

$$\rho(\Gamma(R, t, v, q, s)) = \xi(\Upsilon(v, q, s)) \rho(A(t)) \rho(R). \quad (80)$$

The σ representations of $A(t) \in \mathcal{A}(1)$ are simply $\sigma(A(t)) = e^{\frac{i}{\hbar} t \sigma'(E)} = e^{\frac{i}{\hbar} t \varepsilon}$ with $t, \varepsilon \in \mathbb{R}$. We can then put everything together, as in the Hamilton group case, to obtain the faithful unitary irreducible representations

$$\begin{aligned} \varphi'(\tilde{p}) &= (\varrho(\Gamma(R, t, v, q, s)) \varphi)(\tilde{p}) \\ &= (\sigma(R) \sigma(A(t)) \otimes \xi(\Upsilon(v, q, s)) \rho(A(t)) \rho(R)) \varphi(\tilde{p}) \\ &= \sigma(R) e^{\frac{i}{\hbar} t \varepsilon} \otimes e^{\frac{i}{\hbar} (\mu s + q \cdot p + \frac{1}{2\mu} t p^2)} \varphi(R^{-1} \tilde{p} - \mu v). \end{aligned} \quad (81)$$

In this expression, $v, q, p \in \mathbb{R}^n$ and $s, \mu \in \mathbb{R}$, $\mu \neq 0$ and $\varphi \in \mathbb{V}^N \otimes \mathbf{L}^2(\mathbb{R}^n, \mathbb{C})$. Again, for $n = 3$, $N = 2j + 1$ and the σ representation is given in terms of the usual D matrices,

$$\varphi'_m(\tilde{p}) = D^j_m(R) e^{\frac{i}{\hbar} (\mu s + q \cdot p + \frac{1}{2\mu} t p^2)} \varphi_m(R^{-1} \tilde{p} - \mu v). \quad (82)$$

This is the same as the well known results for the Galilei group determined from the abelian Mackey theorem (Theorem 8) using the semidirect product form in (31) with $\mathcal{A}(n + 2)$ as the normal subgroup [8].

Appendix D shows how the representation of its cover $\overline{\mathcal{G}a}(n)$ is computed from these results. The degenerate cases in Appendix C (123) may similarly be computed using these methods.

4.3.3. *Casimir invariants.* The Casimir invariants of the Galilei group for $n = 3$ are given in (21). A straightforward calculation shows that $\rho'(C_2) = 0$ and $\rho'(C_3) = 0$. Therefore,

$$\begin{aligned}\varrho'(C_1) &= \rho'(M) = \mu, \\ \varrho'(C_2) &= 2\mu\sigma'(E) = 2\mu\varepsilon, \\ \varrho'(C_3) &= \mu^2\sigma'(J^2) = \mu^2j(j+1).\end{aligned}\tag{83}$$

where $\mu, \varepsilon \in \mathbb{R}$, $\mu \neq 0$ and j half integral. The faithful irreducible representations of the Galilei group are labeled by the eigenvalues (μ, ε, j) . Similar considerations apply to the degenerate cases.

4.4. Projective representations of the inhomogeneous Hamilton group.

The projective representations of the inhomogeneous Hamilton group $\mathcal{IHa}(n)$ are the unitary representations of its central extension $\mathcal{IHa}(n) \simeq \overline{\mathcal{QHa}}(n)$ that was given in Section 3. We undertake the calculation for $\mathcal{QHa}(n)$ and then show in Appendix D how the result for the cover follows. The unitary irreducible representations obtained using the Mackey Theorem 7 consist of the nondegenerate faithful representations and the rich set of degenerate representations that correspond to faithful representations of the homomorphisms of the group given in Appendix C (123). We focus here on the faithful representations and leave the degenerate cases as an exercise for the reader using the same methods.

4.4.1. *The unitary irreducible representations of the normal subgroup.* The normal subgroup of $\mathcal{QHa}(n)$ in the semidirect product (19) is $\mathcal{N} \simeq \mathcal{H}(n+1) \otimes \mathcal{A}(2)$. Its elements are the product of $\Upsilon(q, p, t, \varepsilon, \iota) \in \mathcal{H}(n+1)$, where $q, p \in \mathbb{R}^n$, $t, \varepsilon, \iota \in \mathbb{R}$, and $A(s, u) \in \mathcal{A}(2)$, with $s, u \in \mathbb{R}$. The faithful unitary irreducible representations of the normal subgroup are just the direct product of the unitary irreducible representations of $\mathcal{H}(n+1)$ and $\mathcal{A}(2)$

$$\begin{aligned}\psi' &= \tilde{\xi}(\Upsilon(q, p, t, \varepsilon, \iota))\tilde{\xi}(A(s, u))\psi \\ &= e^{i(\iota\tilde{I} + \frac{1}{\hbar}(q^i\tilde{P}_i + p^i\tilde{Q}_i + t\tilde{E} - \varepsilon\tilde{T}))} e^{i(s\tilde{M} + u\tilde{A})}\psi.\end{aligned}\tag{84}$$

ψ is an element of the Hilbert space $\mathbf{H}^\varepsilon \simeq \mathbf{L}^2(\mathbb{R}^n, \mathbb{C})$ and we denote the hermitian representation of the generators with tilde, $\tilde{Z} = \tilde{\xi}'(Z)$. The generators $\{M, A, I\}$ are central and therefore their hermitian representation is always diagonal.

$$\tilde{I} = \lambda, \quad \tilde{M} = \mu, \quad \tilde{A} = \alpha.\tag{85}$$

Any commuting subset of the hermitian representation of the generators $\{\tilde{Q}_i, \tilde{T}, \tilde{P}_i, \tilde{E}\}$ may be simultaneously diagonalized. Four canonical sets are $\{\tilde{P}_i, \tilde{T}\}$, $\{\tilde{Q}_i, \tilde{T}\}$, $\{\tilde{P}_i, \tilde{E}\}$, and $\{\tilde{Q}_i, \tilde{E}\}$. For example, if we diagonalize $\{\tilde{P}_i, \tilde{T}\}$ the generators are realized in the momentum-time representation, $\psi(\tilde{p}, \tilde{t}) = \langle \tilde{p}, \tilde{t} | \psi \rangle$, as

$$\tilde{P}_i = \tilde{p}_i, \quad \tilde{T} = \lambda\tilde{t}, \quad \tilde{Q}_i = -i\lambda\hbar\frac{\partial}{\partial\tilde{p}_i}, \quad \tilde{E} = i\hbar\frac{\partial}{\partial\tilde{t}}.\tag{86}$$

These satisfy the Heisenberg commutation relations

$$[\tilde{P}_i, \tilde{Q}_i] = i\hbar\tilde{I}\delta_{i,j}, \quad [\tilde{T}, \tilde{E}] = i\hbar\tilde{I}\tag{87}$$

The Weyl-Heisenberg group representation in this basis is

$$\begin{aligned}\psi'(\tilde{p}, \tilde{t}) &= \tilde{\xi}(\Upsilon(q, p, t, \varepsilon, \iota))\tilde{\xi}(A(s, u))\psi(\tilde{p}, \tilde{t}) \\ &= e^{i\vartheta} \psi(\tilde{p} - p, \tilde{t} - t),\end{aligned}\tag{88}$$

where

$$\vartheta = \left(s\mu + u\alpha + \lambda\iota + \frac{1}{\hbar} \left(q \cdot \tilde{p} - \varepsilon\tilde{t} - \frac{\lambda}{2} (q \cdot p - \varepsilon t) \right) \right) \quad (89)$$

with $\mu, \lambda \in \mathbb{R} \setminus \{0\}$ and $\tilde{t} \in \mathbb{R}$, $\tilde{p} \in \mathbb{R}^n$. The physical Heisenberg commutation relations require $\lambda = 1$ and we therefore set $\lambda = 1$ going forward.

4.4.2. *The $\tilde{\rho}$ representations.* The next step is to determine the representation $\tilde{\rho}$ and the stabilizer \mathcal{G}° on which it acts as defined in Theorem 7. (The tilde is to distinguish this ρ representation from the ρ representation of the Hamilton subgroup that we have already determined which will also be required in this calculation.) The algebra of $\mathcal{Ha}(n)$ may be realized in the enveloping algebra of the algebra of $\mathcal{H}(n+1) \otimes \mathcal{A}(2)$. (In this section, the tilde on a generator of the algebra denotes the $\tilde{\rho}'$ representation, $\tilde{Z} = \tilde{\rho}'(Z)$.) Note that $\tilde{\rho}'|_{\mathcal{N}} = \tilde{\xi}'$.

$$\begin{aligned} \tilde{J}_{i,j} &= \frac{1}{\lambda\hbar} \left(\tilde{P}_j \tilde{Q}_i - \tilde{P}_i \tilde{Q}_j \right), \quad \tilde{R} = \frac{1}{2\lambda\hbar} \left(\tilde{T}^2 + \tilde{M} \tilde{A} \right), \\ \tilde{G}_i &= \frac{1}{\lambda\hbar} \left(\tilde{T} \tilde{P}_i - \tilde{M} \tilde{Q}_i \right), \quad \tilde{F}_i = \frac{1}{\lambda\hbar} \left(\tilde{T} \tilde{Q}_i + \tilde{A} \tilde{P}_i \right), \end{aligned} \quad (90)$$

The commutation relations for the generators in the $\tilde{\rho}'$ representation may be directly computed and shown to satisfy the algebra (26-28) of $\mathcal{QHa}(n)$ with the i inserted for the hermitian representation as explained in (41).

As all of the generators of $\mathcal{QHa}(n)$ are realized in this $\tilde{\rho}'$ representation, the stabilizer \mathcal{G}° is the entire group $\mathcal{QHa}(n)$ and the *little* group is $\mathcal{Ha}(n)$. Using the properties of the semidirect product (22), the $\tilde{\rho}$ representation may be written as

$$\tilde{\rho}(\Gamma(R, v, f, r, q, t, p, \varepsilon, \iota, s, u)) = \tilde{\xi}(\Upsilon(q, t, p, \varepsilon, \iota)) \tilde{\xi}(A(s, u)) \tilde{\rho}(\Upsilon(v, f, r)) \tilde{\rho}(R). \quad (91)$$

In the momentum-time representation, the $\{\tilde{J}_{i,j}, \tilde{G}_i, \tilde{F}_i, \tilde{R}\}$ generators are (with $\lambda = 1$)

$$\begin{aligned} \tilde{J}_{i,j} &= i \left(\tilde{p}_i \frac{\partial}{\partial \tilde{p}_j} - \tilde{p}_j \frac{\partial}{\partial \tilde{p}_i} \right), \quad \tilde{R} = \frac{1}{\hbar} \left(\tilde{t}^2 + \mu\alpha \right), \\ \tilde{G}_i &= \frac{1}{\hbar} \left(\tilde{t} \tilde{p}_i + i\hbar\mu \frac{\partial}{\partial \tilde{p}_i} \right), \quad \tilde{F}_i = \frac{1}{\hbar} \left(\alpha \tilde{p}_i - i\hbar\tilde{t} \frac{\partial}{\partial \tilde{p}_i} \right). \end{aligned} \quad (92)$$

The $\tilde{\rho}$ representation of the $\mathcal{SO}(n)$ subgroup with elements R is

$$\psi'(\tilde{p}, \tilde{t}) = (\tilde{\rho}(R)\psi)(\tilde{p}, \tilde{t}) = e^{i\theta^{i,j} \tilde{J}_{i,j}} \psi(\tilde{p}, \tilde{t}) = \psi(R^{-1}\tilde{p}, \tilde{t}). \quad (93)$$

For the $\tilde{\rho}$ representation of the Weyl-Heisenberg subgroup with elements $\Upsilon(v, f, r)$, first note that a general element of this algebra is

$$\begin{aligned} \tilde{Z} &= r\tilde{R} + v^i \tilde{G}_i + f^i \tilde{F}_i \\ &= \frac{1}{\hbar} \left(r(\tilde{t}^2 + \mu\alpha) + (v^i \tilde{t} + f^i \alpha) \tilde{p}_i + (\mu v^i - \tilde{t} f^i) \left(i\hbar \frac{\partial}{\partial \tilde{p}_i} \right) \right). \end{aligned} \quad (94)$$

Therefore,

$$\begin{aligned} \psi'(\tilde{p}, \tilde{t}) &= (\tilde{\rho}(\Upsilon(v, f, r))\psi)(\tilde{p}, \tilde{t}) \\ &= e^{i(v^i \tilde{G}_i + f^i \tilde{F}_i + r\tilde{R})} \psi(\tilde{p}, \tilde{t}) \\ &= e^{i\vartheta'} \psi(\tilde{p} - \mu v + \tilde{t} f, \tilde{t}) \end{aligned} \quad (95)$$

where

$$\vartheta' = \frac{1}{\hbar} \left(r(\tilde{t}^2 + \alpha\mu - \frac{1}{2}(\mu v - \tilde{t} f) \cdot (v\tilde{t} + \alpha f)) + (v\tilde{t} + \alpha f) \cdot \tilde{p} \right). \quad (96)$$

Putting together these equations with (88) gives the full nondegenerate $\tilde{\rho}$ representation of the group in the momentum-time diagonal basis

$$\begin{aligned}\psi'(\tilde{p}, \tilde{t}) &= (\tilde{\rho}(\Gamma(R, v, f, r, q, t, p, \varepsilon, \iota, s, u))\psi)(\tilde{p}, \tilde{t}) \\ &= e^{i(\vartheta+\vartheta')}\psi(R^{-1}\tilde{p} - \mu v + tf - p, \tilde{t} - t).\end{aligned}\quad (97)$$

A similar calculation shows that in a position time basis, this results in

$$\begin{aligned}\psi'(\tilde{q}, \tilde{t}) &= (\tilde{\rho}(\Gamma(R, v, f, r, q, t, p, \varepsilon, \iota, s, u))\psi)(\tilde{q}, \tilde{t}) \\ &= e^{\frac{i}{\hbar}(\vartheta+\vartheta')}\psi(R^{-1}\tilde{q} + \alpha f + tv - q, \tilde{t} - t)\end{aligned}\quad (98)$$

where in this expression

$$\begin{aligned}\tilde{\vartheta} &= s\mu + u\alpha + \iota + \frac{1}{\hbar}(p \cdot \tilde{q} - \varepsilon\tilde{t} + \frac{1}{2}(q \cdot p - \varepsilon t)), \\ \tilde{\vartheta}' &= \frac{1}{\hbar}(r(\tilde{t}^2 + \alpha\mu + \frac{1}{2}(\mu v - \tilde{t}f) \cdot (v\tilde{t} + \alpha f)) + (\mu v - \tilde{t}f) \cdot \tilde{q}).\end{aligned}\quad (99)$$

4.4.3. Nondegenerate unitary irreducible representations. As the stabilizer is the entire group $\mathcal{G} \simeq \mathcal{QHa}(n)$, the Mackey induced representation theorem (Theorem 6) is not required and the unitary irreducible representations are given by

$$\varrho(\Gamma) = \tilde{\sigma}(K) \otimes \tilde{\rho}(\Gamma) = \sigma(R) \otimes \rho(K) \otimes \tilde{\rho}(\Gamma), \quad (100)$$

with $R \in \mathcal{SO}(n)$, $K \in \mathcal{K}^\circ \simeq \mathcal{Ha}(n)$ and $\Gamma \in \mathcal{QHa}(n)$. The $\tilde{\sigma}$ representations are the unitary irreducible representations of the Hamilton group that are referred to as the ϱ representations in (72), $\tilde{\sigma}(K) \simeq \sigma(R) \otimes \rho(K)$. The $\tilde{\rho}$ representations are given above in (97).

Putting it all together, for $n = 3$ the nondegenerate unitary irreducible representation of $\mathcal{QHa}(3)$ in a basis with $\{G_i, P_i, T\}$ diagonal is

$$\psi'_{\tilde{m}, \tilde{f}}(\tilde{p}, \tilde{t}) = e^{i\vartheta'''} D^j(R)_{\tilde{m}}^m \psi_{m, R^{-1}\tilde{f}-f}(R^{-1}\tilde{p} - \mu v + f\tilde{t} - p, \tilde{t} - t), \quad (101)$$

where we have set $\lambda = 1$ and the phase ϑ''' is

$$\vartheta''' = \vartheta + \vartheta' + \vartheta'', \quad \vartheta'' = \kappa(r - \frac{1}{2}v \cdot f + v \cdot \tilde{f}).$$

ϑ'' is the phase of the ρ representation of the Hamilton subgroup that is given in (72).

One could also choose to have $\{F_i, P_i, T\}$ to be diagonal. Using (73), the position-time wave functions are

$$\tilde{\psi}'_{\tilde{m}, \tilde{v}}(\tilde{q}, \tilde{t}) = e^{i\tilde{\vartheta}'''} D^j(R)_{\tilde{m}}^m \tilde{\psi}_{m, R^{-1}\tilde{v}-v}(R^{-1}\tilde{q} - v\tilde{t} - \alpha f - q, \tilde{t} - t). \quad (102)$$

where we have set $\lambda = 1$ the phase $\tilde{\vartheta}'''$ is (73)

$$\tilde{\vartheta}''' = \vartheta + \vartheta' + \tilde{\vartheta}'', \quad \tilde{\vartheta}'' = \kappa(r + \frac{1}{2}v \cdot f + f \cdot \tilde{v}).$$

Other combinations of generators that can be simultaneously diagonalized include $\{G_i, Q_i, T\}$ and $\{F_i, Q_i, T\}$. Their representations follow from (98) together with (72) and (73). The Hilbert space for all of these representations is

$$\mathbf{H}^e = \mathbb{V}^{2j+1} \otimes \mathbf{L}^2(\mathbb{R}^n, \mathbb{C}) \otimes \mathbf{L}^2(\mathbb{R}^{n+1}, \mathbb{C}). \quad (103)$$

The corresponding representation of the Lie algebra is

$$\begin{aligned}\varrho'(J_{i,j}) &= \sigma'(J_{i,j}) + \hat{J}_{i,j} + \tilde{J}_{i,j}, \\ \varrho'(\{G_i, F_i, R\}) &= \left\{ \hat{G}_i + \tilde{G}_i, \hat{F}_i + \tilde{F}_i, \hat{R} + \tilde{R} \right\}, \\ \varrho'(\{Q_i, P_i, T, E, I, M, A\}) &= \left\{ \tilde{Q}_i, \tilde{P}_i, \tilde{T}, \tilde{E}, \tilde{I}, \tilde{M}, \tilde{A} \right\}.\end{aligned}\quad (104)$$

Appendix D shows how the representation of the cover $\overline{\mathcal{QHa}}(n)$ is computed from these results.

4.4.4. *Casimir invariants.* The Casimir invariants are given in (16) for the case $n = 3$. A straightforward calculation shows that $\tilde{\rho}'(C_4) = 0$ and $\tilde{\rho}'(C_5) = 0$. Combining this with the corresponding results for the Hamilton group that is the homogeneous group, the representations of these Casimirs are

$$\begin{aligned} \varrho'(C_1) &= \tilde{\rho}'(I) = \lambda, & \varrho'(C_2) &= \tilde{\rho}'(M) = \mu, & \tilde{\rho}'(C_3) &= \tilde{\rho}'(A) = \alpha, \\ \varrho'(C_4) &= \tilde{\rho}'(I)\rho'(R) - \tilde{\rho}'(MA) = \kappa\lambda - \mu\alpha, \\ \varrho'(C_5) &= (\alpha\mu - \kappa\lambda)^2 j(j+1). \end{aligned} \tag{105}$$

Thus the $(\lambda, \mu, \alpha, \kappa, j)$ label the nondegenerate projective representations of the quantum mechanical Hamilton group for $n = 3$. Again, the physical Heisenberg commutation relations correspond to the irreducible representation with $\lambda = 1$

5. DISCUSSION

This paper started with the observation that a most basic feature of quantum mechanics is that physical states are rays that are equivalence classes of states in a Hilbert space defined up to a phase. A quote from Dirac later in his life exemplifies this. "So if one asks what is the main feature of quantum mechanics, I feel inclined now to say that it is not noncommutative algebra. It is the existence of probability amplitudes which underlie all atomic processes. Now a probability amplitude is related to experiment but only partially. The square of the modulus is something that we can observe. That is the probability which the experimental people get. But besides that there is a phase, a number of modulus unity which we can modify without affecting the square of the modulus. And this phase is all important because it is the source of all interference phenomena but its physical significance is obscure." [9]

This physical requirement that the states in quantum mechanics are rays requires projective representations of symmetry groups in quantum mechanics rather than ordinary unitary representations. The projective representations have the remarkable property that a set of theorem that enables us to compute the representations for a general class of connected Lie groups. The projective representations of the groups discussed in the paper illustrate the power of these theorems and these methods can be directly applied to a other connected Lie groups.¹² First, the cornerstone Theorem 1 states that a projective representation of a connected group is equivalent to a projective representation that is unitary and linear. Theorem 2 state further that a projective representations are equivalent to the unitary representations of its central extension. Central extensions are simply connected and therefore Levi's Theorem 4 states that they are equivalent to a semidirect product of a semi-simple subgroup and a solvable normal subgroup. The unitary representations of the semi-simple groups are generally known and the solvable group in the applications we encounter turn out to be semidirect products of abelian groups. We have shown how to compute the irreducible representations of the solvable Weyl-Heisenberg group, and the $\mathcal{Ga}(n)$, $\mathcal{Ha}(n)$ and $\mathcal{QHa}(n)$ groups that have it as a normal subgroup, using the Mackey Theorems 6-8.

¹²These method's may also be applied to Lie groups that are not connected but in this case the central extension may not be unique and therefore they must be addressed on a case by case basis.

Furthermore, the form of the semidirect product is constrained by the automorphism Theorem 5. We expect any physical symmetry to leave invariant the Heisenberg commutation relations. That is, under a symmetry transformation on phase space must result in

$$i\hbar\delta_{i,j} = [\hat{P}_i, \hat{Q}_j] = \varrho(g)[\hat{P}_i, \hat{Q}_j]\varrho(g)^{-1} = [\widehat{P}'_i, \widehat{Q}'_j], \quad (106)$$

with $\widehat{P}'_i = \varrho(g)\hat{P}_i\varrho(g)^{-1}$ and $\widehat{Q}'_i = \varrho(g)\hat{Q}_i\varrho(g)^{-1}$ where g is an element of the symmetry group \mathcal{G} . Similar considerations hold for the energy-time commutation relations. This requires \mathcal{G} to be a subgroup of the automorphism group of the Weyl-Heisenberg group. Therefore, the maximal symmetry is $\mathcal{D}Sp(2n+2) \simeq \mathcal{D} \otimes \mathcal{ISp}(2n+2)$ as the central extension of this group is the Weyl-Heisenberg automorphism group. Furthermore, the projective representation of this maximal symmetry results in the Heisenberg commutation relations and the Hilbert space for these operators.

These mathematical theorems immediately give us the result that the quantum mechanical phase leads directly to the noncommutative algebra of quantum mechanics as stated by Dirac in the quote at the beginning of this section. The symplectic homogenous group of $\mathcal{D}Sp(2n)$ constrains the central extension of the abelian subgroup $\mathcal{A}(2n+2)$ to be the Weyl-Heisenberg group and therefore particular projective representations of this abelian group are the unitary representations of the Weyl-Heisenberg group. The physical meaning of the abelian group is translations on extended phase space $\mathbb{P} \simeq \mathbb{R}^{2n+2}$ and the hermitian representation of the Lie algebra corresponding to the unitary representations of the Weyl-Heisenberg group are the Heisenberg commutation relations.

Neither the Weyl-Heisenberg group nor the symplectic group have any concept of an invariant time line element. This is a world before any relativistic structure is present. There is no notion of null cones, past and future, or causality.

The relativistic structures of invariant time and mass line elements may then be defined,

$$d\tau^2 = dt^2 - \frac{1}{c^2}dq^2, \quad d\mu^2 = \frac{1}{c^2}d\varepsilon^2 - dp^2. \quad (107)$$

These may be regarded as two degenerate line elements on the cotangent of extended phase space, $T^*\mathbb{P}$. This now differentiates key properties of time and energy from position and momentum. They introduce the concepts of null cones and the notions of past and future and causality. The invariance of these line elements requires the subgroup $\mathcal{L}(1,n)^{13}$ of $Sp(2n+2)$. The invariance of both of these line elements results in inertial states. This allows the phase space to be *broken apart* into space-time and energy-momentum space as this symmetry does not mix these degrees of freedom. As we have noted in the introduction, special relativistic quantum mechanics results from the projective representations of $\mathcal{IL}(1,n)$.

However, not all physical states are inertial. This leads us to follow Born [21],[22] and combine the invariants into a single nondegenerate invariant line element for time

$$d\tilde{\tau}^2 = dt^2 - \frac{1}{c^2}dq^2 - \frac{1}{b^2}dp^2 + \frac{1}{b^2c^2}d\varepsilon^2. \quad (108)$$

¹³ $\mathcal{L}(1,n)$ is the connected subgroup of $\mathcal{O}(1,n)$ that is the full symmetry.

Requiring invariance of this line element results in the subgroup $\mathcal{U}(1, n)$ of $Sp(2n + 2)$. $\mathcal{L}(1, n)$ is the inertial subgroup of $\mathcal{U}(1, n)$. The constants $\{c, b, \hbar\}$ define the dimensional basis where b , that has dimensions of force, is a bound on the rate of change of momentum just as c is a bound on the rate of change of position. In this basis, gravitational coupling is defined by the dimensionless constant α_G where $G = \alpha_G c^4 / b$. As G and c are known, determining α_G or b defines the other. These effects are manifest only for forces approaching b , which if α_G is anywhere near unity, is very large.

A noninertial relativistic quantum theory results from the projective representations of $\mathcal{IU}(1, n)$ with this definition of invariant time that includes inertial and noninertial states [23, 24, 25, 26, 27]. These are given by the unitary representations of its central extension $\overline{\mathcal{Q}}(1, n) \simeq \overline{\mathcal{U}}(1, n) \otimes_s \mathcal{H}(n + 1)$. The hermitian algebra of the unitary representations of $\mathcal{H}(n + 1)$ are the Heisenberg commutation relations.

The limits of the line elements for small forces relative to b and small velocity relative to c are

$$d\tilde{\tau}^2 \xrightarrow{b \rightarrow \infty} d\tau^2 \xrightarrow{c \rightarrow \infty} dt^2, \quad (109)$$

with corresponding Inönü-Wigner group contractions [28, 29]

$$\mathcal{IU}(1, n) \xrightarrow{b \rightarrow \infty} \mathcal{IOa}(1, n) \xrightarrow{c \rightarrow \infty} \mathcal{IHa}(1, n). \quad (110)$$

The group $\mathcal{IOa}(1, n)$ and its projective representations is studied in [29]. There exists a homomorphism of $\mathcal{IOa}(1, n)$ onto $\mathcal{IL}(1, n)$ and therefore the usual representations of special relativistic quantum mechanics are a degenerate representation.

In this paper, we are focussed on the full $b, c \rightarrow \infty$ limit group $\mathcal{IHa}(1, n)$ that is the symmetry in the *nonrelativistic* regime [11], [16], [17]. The projective representations are the unitary representations of its central extension $\overline{\mathcal{QHa}}(n)$.

The first summary observation is that the projective representations of $\mathcal{IHa}(1, n)$ contains precisely the unitary representations of a Weyl-Heisenberg subgroup for which its algebra is the physical Heisenberg commutation relations as is also the case for both the $\mathcal{IU}(1, n)$ and $\mathcal{IOa}(1, n)$ groups. Simply by considering projective representations of the symmetry $\mathcal{IHa}(n)$ on phase space, we obtain the noncommutative algebra of quantum mechanics. Again, just as stated by Dirac in the quote at the beginning of this section, the noncommutative structure arises because of the existence of the quantum phase. All of these symmetry have an abelian subgroup of translations on extended phase space that is parameterized by position, time, momentum and energy degrees of freedom and yet result in the expected wave functions that are functions of the eigenvalues of commuting subsets. That is we, obtain wave functions of the form $\psi(p, t)$ (101) and $\psi(q, t)$ (102), and not $\psi(q, p, e, t)$ as, for example, is given by Wigner [30], [31] or Moyal [32] in their phase space formulations of quantum mechanics. The Fourier transform between position and momentum representations is just the unitary representation of the isomorphism on the Weyl-Heisenberg group (46).

The central extension also contains the central generator M that is mass. This generator appears in precisely the form required for the Galilei group to be the inertial subgroup. This central extension embodies energy, Mc^2 . The Galilei group $\overline{\mathcal{Ga}}(n)$ is both a subgroup and a group homomorphic to $\overline{\mathcal{QHa}}(n)$ as given in Appendix C (123). A consequence of Theorem 3 is that the complete set of

unitary irreducible representations of $\overline{\mathcal{QHa}}(n)$ includes the faithful unitary irreducible representations of all of these homomorphic groups as degenerate representations. Therefore, the unitary irreducible representations of $\overline{\mathcal{QHa}}(n)$ include the usual unitary irreducible representations of $\overline{\mathcal{Ga}}(n)$ corresponding to the inertial states. However, the quantum mechanical Hamilton representations are on a larger Hilbert space and include noninertial states not present in the representations of the Galilei group.

The group $\overline{\mathcal{QHa}}(n)$ has a third central generator A that has physical dimensions of the reciprocal of tension. Studying the limits (110) carefully shows that it also embodies energy, Ab^2 just as mass embodies energy, Mc^2 . A is as fundamental to the physical interpretation of this mathematical theory as M and I . The latter two are clearly very fundamental and so if this symmetry has physical relevance, A must also be fundamental. This is a definitive test of the overall symmetry - either physical phenomena corresponding to the presence of A must be found to exist or an explanation of why its eigenvalue α is zero must be provided. At this point, we can only note that, as it embodies energy it will gravitate, and we know that we are missing a lot of gravitational “mass” and “energy” in the current standard theory that we refer to as dark.

The expression (101) for the unitary irreducible representations of $\mathcal{QHa}(n)$ may be written as

$$\psi'_{\tilde{m}, \tilde{f}'}(\tilde{p}', \tilde{t}') = e^{i\vartheta'''} D^j(R)_{\tilde{m}}^m \psi_{m, \tilde{f}}(\tilde{p}, \tilde{t}) \quad (111)$$

where

$$\tilde{p}' = R(\tilde{p}' + \mu v - f\tilde{t} + p), \tilde{t}' = \tilde{t} + t, \tilde{f}' = R(\tilde{f} + f). \quad (112)$$

These transformations of momentum and time are of the expected form if force is constant. This is because the symmetry is global in the treatment that we have provided. However, in general, we expect these transformations to be local [11]. That is,

$$d\tilde{p}' = R(d\tilde{p}' + v(p, q, t)d\mu - f(p, q, t)d\tilde{t} + dp). \quad (113)$$

This will require the symmetry group to be gauged so that it is local before attempting a full physical interpretation. This will be the topic of a subsequent paper.

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APPENDIX A. PROJECTIVE REPRESENTATIONS AND CENTRAL EXTENSIONS

This appendix shows how Theorem 2 results in the understanding of projective representations as a unitary representation up to a phase. Consider a simply connected group $\overline{\mathcal{G}}$ with central extension $\check{\mathcal{G}}$ where the central extension includes a nontrivial algebraic central extension. The algebra of $\check{\mathcal{G}}$ is spanned by the generators $\{X_a, A_\alpha\}$ where the A_α are central, $[X_a, A_\alpha] = 0$ and $[A_\beta, A_\alpha] = 0$. As $\check{\mathcal{G}}$ is always simply connected, an element $\Gamma \in \check{\mathcal{G}}$ may be expressed as

$$\Gamma(x, a) = e^{a^\alpha A_\alpha} \gamma(x) \quad , \quad \gamma(x) = e^{x^a X_a}, \quad (114)$$

where $x \in \mathbb{R}^r, a \in \mathbb{R}^m$ with $n = r + m$. The $\mathcal{A}(m)$ is the central subgroup of $\check{\mathcal{G}}$ with an algebra spanned by A_a . Using the Baker-Campbell-Hausdorff formula, the group product is,

$$\begin{aligned}\Gamma'(x', a')\Gamma(x, a) &= e^{(a'^\alpha + a^\alpha)A_\alpha} e^{\beta(x', x)^a X_a + \alpha(x', x)^\alpha A_\alpha} \\ &= e^{(a'^\alpha + a^\alpha + \alpha(x', x)^\alpha)A_\alpha} \gamma(x''),\end{aligned}\quad (115)$$

where $x'' = \beta(x', x)$. Furthermore, the terms that define β are precisely the Baker-Campbell-Hausdorff terms that result from the group product “ \cdot ” of $\check{\mathcal{G}}$, $\gamma(x'') = \gamma(x') \cdot \gamma(x)$. Therefore, the group product may be written as

$$\Gamma'(x', a')\Gamma(x, a) = e^{(a'^\alpha + a^\alpha + \alpha(x', x)^\alpha)A_\alpha} \gamma(x') \cdot \gamma(x). \quad (116)$$

In this sense, the elements $\gamma(x)$ with the multiplication \cdot may be regarded to be elements of $\check{\mathcal{G}}$. Note that $\check{\mathcal{G}}$ is not necessarily a subgroup of $\check{\mathcal{G}}$ as the α depend on x and x' . The map $\alpha : \check{\mathcal{G}} \times \check{\mathcal{G}} \rightarrow \mathcal{A}(m)$ may be shown to be an element of the second cohomology group $H^2(\mathcal{G}, \mathcal{A}(m))$.

Theorem 1 states that every projective representation of any connected Lie group is equivalent to a projective representation that is unitary and therefore, up to an equivalence, we need only consider unitary representations. The unitary representations of elements $\Gamma(x, a)$ of $\check{\mathcal{G}}$ may be written as $\varrho(\Gamma) = \varrho(\gamma(x)) e^{ia \cdot \nu}$ with $\nu \in \mathbb{R}^m$. The unitary representation of the abelian central subgroup $\mathcal{A}(m)$ is the phase $e^{ia \cdot \nu}$. The unitary representations of this group product is

$$\varrho(\Gamma')\varrho(\Gamma) = e^{i(a' + a + \alpha(x', x)) \cdot \nu} \varrho(\gamma'(x')) \cdot \varrho(\gamma(x)). \quad (117)$$

This is a projective representation defined as a unitary representations of $\check{\mathcal{G}}$ up to a continuous phase defined by the continuous central group $\mathcal{A}(m)$. Conversely, given an expression of the form (119) where the elements $\alpha(x', x)$ are elements of the second cohomology group, then this is the unitary representation of a central extension (See section 2.7 of [4]. If \mathcal{G} is simply connected so that $\mathcal{G} \simeq \check{\mathcal{G}}$, then this is the maximal set of phases that can be constructed.

If \mathcal{G} is not simply connected, we must consider the topological central extension resulting from a nontrivial homotopy. As $\check{\mathcal{G}}$ is simply connected, its homotopy group is trivial and any loop can be continuously deformed into the null loop. The kernel of the projection $\pi^\circ : \check{\mathcal{G}} \rightarrow \mathcal{G} : \gamma \mapsto \tilde{\gamma}$ is the homotopy group $\ker \pi^\circ \simeq \mathbb{A} \subset \check{\mathcal{G}}$ that is a discrete central subgroup. The continuous curves

$$\gamma : [0, 1] \rightarrow \check{\mathcal{G}}, \quad \gamma_a(0) = e, \quad \gamma_a(1) = a, \quad a \in \mathbb{A}, \quad (118)$$

project onto loops in \mathcal{G} , as $\pi^\circ(a) = e$, that are representatives of the homotopy classes. If there is a point $t_1 \in [0, 1]$ such that $\gamma(t_1) = a_1$, then the projected loop winds twice, and if there are $n-1$ such points, $\gamma(t_i) = a_i$ the projected loop winds n times. If we consider the case $\gamma(t) = \gamma_1(t)\gamma_2(t)\gamma_3(t)^{-1}$ with $\gamma_i(0) = e, \gamma_i(1) = \gamma_i$, then this has the property that $\gamma_1\gamma_2\gamma_3^{-1} = a$ projects onto a loop where $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3^{-1} = e$. Thus the product $\tilde{\gamma}_1\tilde{\gamma}_2 = \tilde{\gamma}_3$ is associated with a homotopy class that in the covering group is just

$$\gamma_1\gamma_2 = a\gamma_3. \quad (119)$$

The unitary representation of (119) is $\varrho(\gamma_1)\varrho(\gamma_2) = e^{i\phi}\varrho(\gamma_3)$ where $e^{i\phi} = \varrho(a)$ are the discrete phases that are the unitary representations of the discrete center \mathbb{A} . This again has the form of a unitary representation up to a phase and is the maximal set arising from a discrete central group. However, these phases due to

the homotopy cannot be realized locally in terms of the algebra. This is a nonlocal effect and it does not appear in the local expression for the *representation up to a phase* given in (117).

As the full central extension has a maximal center $\mathcal{Z} \simeq \mathbb{A} \otimes \mathcal{A}(m)$, these are the maximal set of phases that can be so constructed.

APPENDIX B. MATRIX REALIZATION OF $\mathcal{QHa}(n)$

The group $\mathcal{QHa}(n)$ is a matrix group with elements $\Gamma(R, v, f, r, t, q, p, \varepsilon, s, u, \iota)$ realized by the $2n + 6$ dimensional matrices

$$\begin{pmatrix} R & 0 & 0 & f & 0 & 0 & 0 & p \\ 0 & R & 0 & v & 0 & 0 & 0 & q \\ v^t R & -f^t R & 1 & r & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & t \\ 0 & v^t R & 0 & v^2/2 & 1 & 0 & 0 & s \\ f^t R & 0 & 0 & f^2/2 & 0 & 1 & 0 & u \\ (q - tv)^t R & -(p - tf)^t R & -t & \tilde{\varepsilon} & 0 & 0 & 1 & 2\iota \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (120)$$

where $\tilde{\varepsilon} = \varepsilon - rt + q^t R f - p^t R v$.

There is no direct algorithm to find matrix groups. However, once the matrices realizing the group elements are found, it is straightforward to establish that they indeed are a matrix group. The above realization was inferred by first noting that $\mathcal{Ha}(n) \otimes_s \mathcal{H}(n+1) \subset \mathcal{HSp}(2n+2)$ where

$$\mathcal{HSp}(2n+2) \simeq \mathcal{Sp}(2n+2) \otimes_s \mathcal{H}(n+1). \quad (121)$$

$\mathcal{HSp}(2m)$ is known to be a matrix group with elements that are $2m+2$ dimensional matrices of the form

$$\begin{pmatrix} A & 0 & w \\ 0 & 1 & \iota \\ -w^t \cdot \zeta \cdot A & 0 & 1 \end{pmatrix}, \quad A \in \mathcal{Sp}(2m), \quad w \in \mathbb{R}^m, \quad \iota \in \mathbb{R}, \quad (122)$$

where ζ is a symplectic matrix. Furthermore, the matrix realization of the Galilei group has elements that are $n+3$ dimensional matrices with the well known form

$$\begin{pmatrix} R & v & 0 & q \\ 0 & 1 & 0 & t \\ v^t R & v^2/2 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R \in \mathcal{SO}(n), \quad v, q \in \mathbb{R}^n, \quad t, s \in \mathbb{R}. \quad (123)$$

These two facts enable us to deduce the above realization of $\mathcal{QHa}(n)$. It is straightforward to verify that matrix multiplication realizes the group product and inverse (20-21). Furthermore, the derivative at the identity yields a matrix realization of the algebra that satisfies the commutation relations (26-28).

APPENDIX C. HOMOMORPHIC GROUPS AND DEGENERATE REPRESENTATIONS

The following is a table of the groups homomorphic to the Weyl-Heisenberg, Hamilton and Galilei groups. The generators of the homomorphic groups are the complement set of generators. That is, the union of the set of generators that are a basis of the normal subgroup and the set that is a basis of the homomorphic group is a basis for the full group.

Group	Normal Subgroup	Homomorphic Group	Generators of Normal Subgroup
$\mathcal{H}(n)$	$\mathcal{A}(1)$	$\mathcal{A}(2n)$	$\{I\}$
$\overline{\mathcal{H}a}(n)$	$\mathcal{A}(1)$	$\overline{\mathcal{SO}}(n) \otimes_s \mathcal{A}(2n)$	$\{R\}$
	$\mathcal{A}(n+1)$	$\overline{\mathcal{SO}}(n) \otimes_s \mathcal{A}(n)$	$\{G_i, R\}$
	$\mathcal{A}(n+1)$	$\overline{\mathcal{SO}}(n) \otimes_s \mathcal{A}(n)$	$\{F_i, R\}$
	$\mathcal{H}(n)$	$\overline{\mathcal{SO}}(n)$	$\{G_i, F_i, R\}$
$\overline{\mathcal{G}a}(n)$	$\mathcal{A}(1)$	$\overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+1)$	$\{M\}$
	$\mathcal{A}(n+1)$	$(\overline{\mathcal{SO}}(n) \otimes \mathcal{A}(1)) \otimes_s \mathcal{A}(n)$	$\{P_i, M\}$
	$\mathcal{A}(n+2)$	$\overline{\mathcal{E}}(n)$	$\{E, P_i, M\}$
	$\mathcal{H}(n)$	$\overline{\mathcal{SO}}(n) \otimes \mathcal{A}(1)$	$\{G_i, P_i, M\}$
	$\mathcal{A}(1) \otimes_s \mathcal{H}(n)$	$\overline{\mathcal{SO}}(n)$	$\{E, G_i, P_i, M\}$

In addition, there is a homomorphism $\overline{\mathcal{SO}}(n) \rightarrow \mathcal{SO}(n)$ with a kernel that is the normal subgroup \mathbb{Z}_2 . Therefore, for each of the entries above containing an $\overline{\mathcal{SO}}(n)$, there is a corresponding homomorphic group with kernel \mathbb{Z}_2 that has an isomorphic Lie algebra. This defines a corresponding set of homomorphic groups that is the same as the above with the bars denoting the cover removed. These groups are not simply connected and their fundamental homotopy group is \mathbb{Z}_2 . From Theorem 3, these appear as degenerate representations.

The following is a table of the groups homomorphic to $\mathcal{I}\tilde{\mathcal{H}a}(n) \simeq \overline{\mathcal{QH}}(n)$ with a connected normal subgroup.

Normal Subgroup	Homomorphic Group	Generators of Normal Subgroup
$\mathcal{A}(1)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{H}(n+1)$	$\{A\}, \{M\}$
$\mathcal{A}(1)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(2n+2)$	$\{I\}$
$\mathcal{A}(2)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{H}(n+1)$	$\{A, M\}$
$\mathcal{A}(2)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(2n+1)$	$\{A, I\}, \{M, I\}$
$\mathcal{A}(3)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(2n)$	$\{I, A, M\}$
$\mathcal{A}(n+3)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(n+1)$	$\{P_i, T, M, I\}$
$\mathcal{A}(n+3)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(n+1)$	$\{Q_i, T, A, I\}$
$\mathcal{H}(n) \otimes \mathcal{A}(2)$	$\overline{\mathcal{G}a}(n)$	$\{P_i, G_i, R, T, M, I\}$
$\mathcal{H}(n) \otimes \mathcal{A}(2)$	$\overline{\mathcal{G}a}(n)$	$\{Q_i, F_i, R, T, A, I\}$
$\mathcal{A}(n+4)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(n)$	$\{P_i, T, M, A, I\}$
$\mathcal{A}(n+4)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(n)$	$\{Q_i, T, A, M, I\}$
$\mathcal{H}(n+1)$	$\overline{\mathcal{H}a}(n) \otimes_s \mathcal{A}(2)$	$\{I, P_i, Q_i, E, T\}$
$\mathcal{H}(n+1) \otimes \mathcal{A}(2)$	$\overline{\mathcal{H}a}(n)$	$\{I, P_i, Q_i, E, T, A, M\}$
$\mathcal{H}(n+1) \otimes \mathcal{A}(3)$	$\overline{\mathcal{SO}}(n) \otimes \mathcal{A}(2n)$	$\{I, P_i, Q_i, E, T, A, M, R\}$
$\mathcal{H}(n+1) \otimes \mathcal{A}(n+3)$	$\overline{\mathcal{E}}(n)$	$\{I, P_i, Q_i, E, T, A, M, F_i, R\}$
$\mathcal{H}(n+1) \otimes \mathcal{A}(n+3)$	$\overline{\mathcal{E}}(n)$	$\{I, P_i, Q_i, E, T, A, M, G_i, R\}$
$\mathcal{H}(n+1) \otimes \mathcal{H}(n) \otimes \mathcal{A}(2)$	$\overline{\mathcal{SO}}(n)$	$\{I, P_i, Q_i, E, T, A, M, F_i, G_i, R\}$

Again, there are also the homomorphisms resulting from $\overline{\mathcal{SO}}(n) \rightarrow \mathcal{SO}(n)$. Noting Theorem 3, this shows that the projective representations of the inhomogeneous Hamilton group has a rich set of degenerate representations.

APPENDIX D. REPRESENTATIONS OF THE COVER OF THE GROUPS FOR $n = 3$

The $\mathcal{H}a(n)$, $\mathcal{G}a(n)$ and $\mathcal{Q}Ha(n)$ all have a $\mathcal{SO}(n)$ subgroup appearing in the semidirect product. The full central extension requires the cover of these groups that have a corresponding $\overline{\mathcal{SO}}(n)$ subgroup. The manner of obtaining these cases follows the method of determining the cover of the Euclidean group $\mathcal{E}(n)$.

The terms of the form Rx , with $R \in \mathcal{SO}(n)$ realized by n dimensional real orthogonal matrices and $x \in \mathbb{R}^n$ where x is one of v, f, p, q are actually automorphisms of the form

$$\varsigma_{\Gamma(R,0,\dots,0)}\Gamma(1, \dots, x, \dots) = \Gamma(1, \dots, Rx, \dots). \quad (124)$$

For $n = 3$, $\overline{\mathcal{SO}}(3) = SU(2)$ with a 2:1 homomorphism $\pi : SU(2) \rightarrow \mathcal{SO}(3) : \bar{R} \mapsto R$. Elements $\bar{R} \in SU(2)$ are realized by 2 dimensional complex unitary matrices. The automorphism action on x is given by representing the x as the 2 dimensional hermitian matrices $\bar{x} = x^i \sigma_i$ where the σ_i are the Pauli matrices. Then

$$\varsigma_{\Gamma(\bar{R},0,\dots,0)}\Gamma(1, \dots, \bar{x}, \dots) = \Gamma(1, \dots, \bar{R}\bar{x}\bar{R}^{-1}, \dots). \quad (125)$$

Substituting $\bar{R}\bar{x}\bar{R}^{-1}$ into the expressions where Rx appears and allowing the j labeling the D^j matrices to take half integral values gives the representation for the cover of the groups.

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S. G. LOW, WWW.STEPHEN-LOW.NET
E-mail address: stephen.low@alumni.utexas.edu

P. D. JARVIS, SCHOOL OF MATHEMATICS AND PHYSICS, UNIVERSITY OF TASMANIA
E-mail address: Peter.Jarvis@utas.edu.au

R. CAMPOAMOR-STURSBURG, I.M.I-U.C.M
Current address: I.M.I-U.C.M, Plaza de Ciencias 3, E-28040 Madrid, Spain
E-mail address: otto_campoamor@mat.ucm.es